International Journal of Theoretical Physics, Vol. 41, No. 6, June 2002 (© 2002)

Synthetic Vector Analysis

Hirokazu Nishimura[1]

Received October 22, 2001

Classical vector analysis is rife with geometric and physical ideas, but appears precarious from a modern viewpoint of pure mathematics. Modern vector analysis with differential forms is surely up to the contemporary standard of mathematical rigor, but geometric ideas are completely lost in the bulk of dull calculations. The main goal in this paper is to show that synthetic differential geometry, which has replenished differential geometry with nilpotent infinitesimals, can mathematically sanitize classical vector analysis by eradicating its total confusion between approximate calculations and infinitesimal calculations, thereby helping it retrieve mathematical rigor.

KEY WORDS: synthetic differential geometry; vector analysis; rotation; divergence; divergence theorem; Stokes's theorem.

1. INTRODUCTION

Vector analysis is a must for any student who majors in physics and its related disciplines. It is an indispensable language for classical mechanics, electromagnetics, and fluid dynamics, to say the least. On the other hand it is a very difficult subject for pure mathematicians to teach [cf. Hirota (2000)]. Its classical treatment is replete with geometric intuition and interest, but is mathematically porous. Its modern treatment with differential forms is mathematically rigorous but obscures the geometric meaning of such important concepts and results as the divergence theorem and Stokes's theorem, possibly causing the subject to degenerate into a mass of dull calculations. The classical treatment of vector analysis lacks mathematical rigor, because it confuses infinitesimal calculations with approximate calculations. It could not help doing so in order to express its geometrically vivid ideas, for there has been no sanctuary for nilpotent infinitesimals in orthodox differential geometry of our age. If we have to call the modern treatment of vector analysis a victory, we would like to call it a Pyrrhic victory.

Synthetic differential geometry enunciated by Lawvere in the 1960s is the vanguard of modern differential geometry, in which nilpotent infinitesimals, once

[1] Institute of Mathematics, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan; e-mail: logic@math.tsukuba.ac.jp.

0020-7748/02/0600-1165/0

ostracized from orthodox differential geometry as logical anathema, are not only abundantly available but also coherently organized with mathematical rigor. The principal objective of this paper is to give a good introduction to synthetic differential geometry for working physicists by presenting a synthetic treatment of classical vector analysis. The paper does not assume any familiarity with the theory of smooth manifolds. We hope that the paper will transmogrify the most conservative physicist into an aficionado of synthetic differential geometry. After reading the paper, the reader is expected to tackle Kock's bible of synthetic differential geometry, which may appear more metaphysical than mathematical at first reading (Kock, 1981). In particular, the reader is highly encouraged to read his geometric treatment of differential forms. The synthetic treatment of differential forms, which is faithful to the tradition of classical vector analysis in spirit, is much more geometric than the standard one.

The organization of this paper goes as follows: Sections 3 and 4 of this paper are devoted to the divergence and Stokes's theorems respectively. Both theorems are proved on the infinitesimal horizon at first, and are generalized to the local level. In Section 5 we are concerned with the geometric meaning of $\mathrm{div} \circ \mathrm{rot} = 0$. Section 2 is a laconic review of elementary calculus from a synthetic viewpoint, though we assume familiarity with Chapter 1 of Lavendhomme (1996).

2. ELEMENTARY CALCULUS

We denote by $\mathbb{R}$ the (extended) set of real numbers with a cornucopia of nilpotent infinitesimals, which is assumed to be a unitary commutative ring. We denote by D the set $\{d \in \mathbb{R} \mid d^2 = 0\}$. The following Kock–Lawvere axiom is enough to engender elementary differential calculus (at least up to Taylor expansions):

Axiom 2.1. *For every function $f : D \to \mathbb{R}$ there exists a unique $b \in \mathbb{R}$ such that $f(d) = f(0) + db$ for any $d \in D$.*

We assume that there exists a preorder $\leq$ on $\mathbb{R}$ which is compatible with the unitary ring structure of $\mathbb{R}$ and satisfies the following condition:

$$0 \leq d \quad \text{and} \quad d \leq 0 \text{ for any } d \in D. \tag{2.1}$$

Given $a, b \in \mathbb{R}$, we denote by $[a, b]$ the set $\{x \in \mathbb{R} \mid a \leq x \leq b\}$.

The following integration axiom is enough to engender elementary integral calculus:

Axiom 2.2. *For every function $f : [0, 1] \to \mathbb{R}$ there exists a unique function $g : [0, 1] \to \mathbb{R}$ such that $g' = f$ and $g(0) = 0$.*

The proof of the following striking proposition is simple.

Proposition 2.1. *Given $a \in \mathbb{R}$, $d \in D$, and a function $f : [a, a] \to \mathbb{R}$, we have*

$$\int_a^{a+d} f(t)\, dt = d f(a). \tag{2.2}$$

Proof: See the proof of Proposition 11 of Lavendhomme (1996, Section 1.3). □

3. DIVERGENCE

Let $\mathbb{f}$ be a vector field on $\mathbb{R}^3$, so that $\mathbb{f}$ assigns $\mathbb{f}(\mathbb{x}) = (f(\mathbb{x}), g(\mathbb{x}), h(\mathbb{x})) = (f(x, y, z), g(x, y, z), h(x, y, z)) \in \mathbb{R}^3$ to each $\mathbb{x} = (x, y, z) \in \mathbb{R}^3$. Let $u_0, v_0, w_0, u_1, v_1, w_1 \in \mathbb{R}$ with $u_0 \leq u_1, v_0 \leq v_1$, and $w_0 \leq w_1$. We assume that we have a function $[u_0, u_1] \times [v_0, v_1] \times [w_0, w_1] \to \mathbb{R}^3$ assigning $\mathbb{x}(\mathbb{u}) = \mathbb{x}(u, v, w) = (x(u, v, w), \ y(u, v, w), \ z(u, v, w)) \in \mathbb{R}^3$ to each $\mathbb{u} = (u, v, w) \in [u_0, u_0] \times [v_0, v_0] \times [w_0, w_0]$. Let $d_1, d_2, d_3 \in D$. Let $\mathbb{u}_0 = (u_0, v_0, w_0)$ and $\mathbb{x}_0 = (x_0, y_0, z_0) = \mathbb{x}(\mathbb{u}_0)$. Then we have

Theorem 3.1 (*Infinitesimal Divergence Theorem*).

$$\begin{aligned}
&\int_{u_0}^{u_0+d_1}\int_{v_0}^{v_0+d_2}\int_{w_0}^{w_0+d_3} (\mathrm{div}\,\mathbb{f})\left[\frac{\partial\mathbb{x}}{\partial u}, \frac{\partial\mathbb{x}}{\partial v}, \frac{\partial\mathbb{x}}{\partial w}\right] du\, dv\, dw \\
&\quad = \int_{u_0}^{u_0+d_1}\int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u, v, w_0 + d_3)) \cdot \left(\frac{\partial\mathbb{x}}{\partial u}(u, v, w_0 + d_3)\right. \\
&\qquad \left. \times \frac{\partial\mathbb{x}}{\partial v}(u, v, w_0 + \ d_3)\right) du\, dv + \int_{v_0}^{v_0+d_2}\int_{w_0}^{w_0+d_3} \mathbb{f}(\mathbb{x}(u_0 + d_1, v, w)) \cdot \\
&\qquad \left(\frac{\partial\mathbb{x}}{\partial u}(u_0 + d_1, v, w) \times \frac{\partial\mathbb{x}}{\partial v}(u_0 + d_1, v, w)\right) dv\, dw + \int_{w_0}^{w_0+d_3}\int_{u_0}^{u_0+d_1} \\
&\qquad \mathbb{f}(\mathbb{x}(u, v_0 + d_2, w)) \cdot \left(\frac{\partial\mathbb{x}}{\partial w}(u, v_0 + d_2, w) \times \frac{\partial\mathbb{x}}{\partial u}(u, v_0 + d_2, w)\right) dw\, du \\
&\qquad - \int_{u_0}^{u_0+d_1}\int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u, v, w_0)) \cdot \left(\frac{\partial\mathbb{x}}{\partial u}(u, v, w_0) \times \frac{\partial\mathbb{x}}{\partial v}(u, v, w_0)\right) du\, dv \\
&\qquad - \int_{v_0}^{v_0+d_2}\int_{w_0}^{w_0+d_3} \mathbb{f}(\mathbb{x}(u_0, v, w)) \cdot \left(\frac{\partial\mathbb{x}}{\partial u}(u_0, v, w) \times \frac{\partial\mathbb{x}}{\partial v}(u_0, v, w)\right) dv\, dw \\
&\qquad - \int_{w_0}^{w_0+d_3}\int_{u_0}^{u_0+d_1} \mathbb{f}(\mathbb{x}(u, v_0, w)) \cdot \left(\frac{\partial\mathbb{x}}{\partial w}(u, v_0, w) \times \frac{\partial\mathbb{x}}{\partial u}(u, v_0, w)\right) dw\, du,
\end{aligned} \tag{3.1}$$

where div $\mathbb{f}$ *stands for* $\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}$, [, ,] *stands for the scalar triple product,* $\cdot$ *stands for the scalar product, and the vector product is denoted by the same symbol* $\times$ *as that for products of sets.*

Proof: We have

$$\int_{u_0}^{u_0+d_1}\int_{v_0}^{v_0+d_2}\mathbb{f}(\mathbb{x}(u,v,w_0))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u,v,w_0)\times\frac{\partial\mathbb{x}}{\partial v}(u,v,w_0)\right)du\,dv$$
$$= d_1d_2\mathbb{f}(\mathbb{x}_0)\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(\mathbb{u}_0)\times\frac{\partial\mathbb{x}}{\partial v}(\mathbb{u}_0)\right)\quad\text{[Proposition 2.1]}\tag{3.2}$$

$$\int_{u_0}^{u_0+d_1}\int_{v_0}^{v_0+d_2}\mathbb{f}(\mathbb{x}(u,v,w_0+d_3))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u,v,w_0+d_3)\right.$$
$$\left.\times\frac{\partial\mathbb{x}}{\partial v}(u,v,w_0+d_3)\right)du\,dv$$
$$= d_1d_2\mathbb{f}(\mathbb{x}(u_0,v_0,w_0+d_3))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u_0,v_0,w_0+d_3)\times\frac{\partial\mathbb{x}}{\partial v}(u_0,v_0,w_0+d_3)\right)$$
$$\text{[Proposition 2.1]}$$
$$= d_1d_2\mathbb{f}\left(\mathbb{x}_0+d_3\frac{\partial\mathbb{x}}{\partial w}(\mathbb{u}_0)\right)\cdot\left\{\left(\frac{\partial\mathbb{x}}{\partial u}(\mathbb{u}_0)+d_3\frac{\partial^2\mathbb{x}}{\partial u\,\partial w}(\mathbb{u}_0)\right)\right.$$
$$\left.\times\left(\frac{\partial\mathbb{x}}{\partial v}(\mathbb{u}_0)+d_3\frac{\partial^2\mathbb{x}}{\partial v\,\partial w}(\mathbb{u}_0)\right)\right\}$$
$$= d_1d_2\left\{\mathbb{f}(\mathbb{x}_0)+d_3\frac{\partial x}{\partial w}(\mathbb{u}_0)\frac{\partial\mathbb{f}}{\partial x}(\mathbb{x}_0)+d_3\frac{\partial y}{\partial w}(\mathbb{u}_0)\frac{\partial\mathbb{f}}{\partial y}(\mathbb{x}_0)+d_3\frac{\partial z}{\partial w}(\mathbb{u}_0)\frac{\partial\mathbb{f}}{\partial z}(\mathbb{x}_0)\right\}\cdot$$
$$\left\{\left(\frac{\partial\mathbb{x}}{\partial u}(\mathbb{u}_0)+d_3\frac{\partial^2\mathbb{x}}{\partial u\,\partial w}(\mathbb{u}_0)\right)\times\left(\frac{\partial\mathbb{x}}{\partial v}(\mathbb{u}_0)+d_3\frac{\partial^2\mathbb{x}}{\partial v\,\partial w}(\mathbb{u}_0)\right)\right\}$$
$$= d_1d_2\mathbb{f}(\mathbb{x}_0)\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(\mathbb{u}_0)\times\frac{\partial\mathbb{x}}{\partial v}(\mathbb{u}_0)\right)+d_1d_2d_3\left\{\mathbb{f}(\mathbb{x}_0)\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(\mathbb{u}_0)\times\frac{\partial^2\mathbb{x}}{\partial v\,\partial w}(\mathbb{u}_0)\right)\right.$$
$$+\mathbb{f}(\mathbb{x}_0)\cdot\left(\frac{\partial\mathbb{x}}{\partial v}(\mathbb{u}_0)\times\frac{\partial^2\mathbb{x}}{\partial u\,\partial w}(\mathbb{u}_0)\right)+\frac{\partial x}{\partial w}(\mathbb{u}_0)\frac{\partial\mathbb{f}}{\partial x}(\mathbb{x}_0)\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(\mathbb{u}_0)\times\frac{\partial\mathbb{x}}{\partial v}(\mathbb{u}_0)\right)$$
$$+\frac{\partial y}{\partial w}(\mathbb{u}_0)\frac{\partial\mathbb{f}}{\partial y}(\mathbb{x}_0)\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(\mathbb{u}_0)\times\frac{\partial\mathbb{x}}{\partial v}(\mathbb{u}_0)\right)+\frac{\partial z}{\partial w}(\mathbb{u}_0)\frac{\partial\mathbb{f}}{\partial z}(\mathbb{x}_0)\cdot$$
$$\left.\left(\frac{\partial\mathbb{x}}{\partial u}(\mathbb{u}_0)\times\frac{\partial\mathbb{x}}{\partial v}(\mathbb{u}_0)\right)\right\}$$

$$
\begin{aligned}
&= d_1 d_2 \mathbb{f}(\mathbb{x}_0) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(\mathbb{u}_0) \times \frac{\partial \mathbb{x}}{\partial v}(\mathbb{u}_0) \right) + d_1 d_2 d_3 \left\{ \mathbb{f}(\mathbb{x}_0) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(\mathbb{u}_0) \times \frac{\partial^2 \mathbb{x}}{\partial v\, \partial w}(\mathbb{u}_0) \right) \right. \\
&+ \mathbb{f}(\mathbb{x}_0) \cdot \left(\frac{\partial \mathbb{x}}{\partial v}(\mathbb{u}_0) \times \frac{\partial^2 \mathbb{x}}{\partial u\, \partial w}(\mathbb{u}_0) \right) + \frac{\partial x}{\partial w}(\mathbb{u}_0) \frac{\partial f}{\partial x}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial y}{\partial u}(\mathbb{u}_0) & \frac{\partial y}{\partial v}(\mathbb{u}_0) \\ \frac{\partial z}{\partial u}(\mathbb{u}_0) & \frac{\partial z}{\partial v}(\mathbb{u}_0) \end{vmatrix} \\
&+ \frac{\partial x}{\partial w}(\mathbb{u}_0) \frac{\partial g}{\partial x}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial z}{\partial u}(\mathbb{u}_0) & \frac{\partial z}{\partial v}(\mathbb{u}_0) \\ \frac{\partial x}{\partial u}(\mathbb{u}_0) & \frac{\partial x}{\partial v}(\mathbb{u}_0) \end{vmatrix} + \frac{\partial x}{\partial w}(\mathbb{u}_0) \frac{\partial h}{\partial x}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial x}{\partial u}(\mathbb{u}_0) & \frac{\partial x}{\partial v}(\mathbb{u}_0) \\ \frac{\partial y}{\partial u}(\mathbb{u}_0) & \frac{\partial y}{\partial v}(\mathbb{u}_0) \end{vmatrix} \\
&+ \frac{\partial y}{\partial w}(\mathbb{u}_0) \frac{\partial f}{\partial y}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial y}{\partial u}(\mathbb{u}_0) & \frac{\partial y}{\partial v}(\mathbb{u}_0) \\ \frac{\partial z}{\partial u}(\mathbb{u}_0) & \frac{\partial z}{\partial v}(\mathbb{u}_0) \end{vmatrix} + \frac{\partial y}{\partial w}(\mathbb{u}_0) \frac{\partial g}{\partial y}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial z}{\partial u}(\mathbb{u}_0) & \frac{\partial z}{\partial v}(\mathbb{u}_0) \\ \frac{\partial x}{\partial u}(\mathbb{u}_0) & \frac{\partial x}{\partial v}(\mathbb{u}_0) \end{vmatrix} \\
&+ \frac{\partial y}{\partial w}(\mathbb{u}_0) \frac{\partial h}{\partial y}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial x}{\partial u}(\mathbb{u}_0) & \frac{\partial x}{\partial v}(\mathbb{u}_0) \\ \frac{\partial y}{\partial u}(\mathbb{u}_0) & \frac{\partial y}{\partial v}(\mathbb{u}_0) \end{vmatrix} + \frac{\partial z}{\partial w}(\mathbb{u}_0) \frac{\partial f}{\partial z}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial y}{\partial u}(\mathbb{u}_0) & \frac{\partial y}{\partial v}(\mathbb{u}_0) \\ \frac{\partial z}{\partial u}(\mathbb{u}_0) & \frac{\partial z}{\partial v}(\mathbb{u}_0) \end{vmatrix} \\
&\left. + \frac{\partial z}{\partial w}(\mathbb{u}_0) \frac{\partial g}{\partial z}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial z}{\partial u}(\mathbb{u}_0) & \frac{\partial z}{\partial v}(\mathbb{u}_0) \\ \frac{\partial x}{\partial u}(\mathbb{u}_0) & \frac{\partial x}{\partial v}(\mathbb{u}_0) \end{vmatrix} + \frac{\partial z}{\partial w}(\mathbb{u}_0) \frac{\partial h}{\partial z}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial x}{\partial u}(\mathbb{u}_0) & \frac{\partial x}{\partial v}(\mathbb{u}_0) \\ \frac{\partial y}{\partial u}(\mathbb{u}_0) & \frac{\partial y}{\partial v}(\mathbb{u}_0) \end{vmatrix} \right\}
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
&\int_{v_0}^{v_0+d_2} \int_{w_0}^{w_0+d_3} \mathbb{f}(\mathbb{x}(u_0, v, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u_0, v, w) \times \frac{\partial \mathbb{x}}{\partial v}(u_0, v, w) \right) dv\, dw \\
&\quad = d_2 d_3 \mathbb{f}(\mathbb{x}_0) \cdot \left(\frac{\partial \mathbb{x}}{\partial v}(\mathbb{u}_0) \times \frac{\partial \mathbb{x}}{\partial w}(\mathbb{u}_0) \right) \quad \text{[Proposition 2.1]}
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
&\int_{v_0}^{v_0+d_2} \int_{w_0}^{w_0+d_3} \mathbb{f}(\mathbb{x}(u_0+d_1, v, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u_0+d_1, v, w) \right. \\
&\left. \times \frac{\partial \mathbb{x}}{\partial v}(u_0+d_1, v, w) \right) dv\, dw
\end{aligned}
$$

$$= d_2 d_3 \mathbb{f}(\mathbb{x}_0) \cdot \left(\frac{\partial \mathbb{x}}{\partial v}(\mathbb{u}_0) \times \frac{\partial \mathbb{x}}{\partial w}(\mathbb{u}_0) \right) + d_1 d_2 d_3 \left\{ \mathbb{f}(\mathbb{x}_0) \cdot \left(\frac{\partial \mathbb{x}}{\partial v}(\mathbb{u}_0) \times \frac{\partial^2 \mathbb{x}}{\partial u\, \partial w}(\mathbb{u}_0) \right) \right.$$

$$+ \mathbb{f}(\mathbb{x}_0) \cdot \left(\frac{\partial \mathbb{x}}{\partial w}(\mathbb{u}_0) \times \frac{\partial^2 \mathbb{x}}{\partial u\, \partial v}(\mathbb{u}_0) \right) + \frac{\partial x}{\partial u}(\mathbb{u}_0) \frac{\partial f}{\partial x}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial y}{\partial v}(\mathbb{u}_0) & \frac{\partial y}{\partial w}(\mathbb{u}_0) \\ \frac{\partial z}{\partial v}(\mathbb{u}_0) & \frac{\partial z}{\partial w}(\mathbb{u}_0) \end{vmatrix}$$

$$+ \frac{\partial x}{\partial u}(\mathbb{u}_0) \frac{\partial g}{\partial x}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial z}{\partial v}(\mathbb{u}_0) & \frac{\partial z}{\partial w}(\mathbb{u}_0) \\ \frac{\partial x}{\partial v}(\mathbb{u}_0) & \frac{\partial x}{\partial w}(\mathbb{u}_0) \end{vmatrix} + \frac{\partial x}{\partial u}(\mathbb{u}_0) \frac{\partial h}{\partial x}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial x}{\partial v}(\mathbb{u}_0) & \frac{\partial x}{\partial w}(\mathbb{u}_0) \\ \frac{\partial y}{\partial v}(\mathbb{u}_0) & \frac{\partial y}{\partial w}(\mathbb{u}_0) \end{vmatrix}$$

$$+ \frac{\partial y}{\partial u}(\mathbb{u}_0) \frac{\partial f}{\partial y}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial y}{\partial v}(\mathbb{u}_0) & \frac{\partial y}{\partial w}(\mathbb{u}_0) \\ \frac{\partial z}{\partial v}(\mathbb{u}_0) & \frac{\partial z}{\partial w}(\mathbb{u}_0) \end{vmatrix} + \frac{\partial y}{\partial u}(\mathbb{u}_0) \frac{\partial g}{\partial y}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial z}{\partial v}(\mathbb{u}_0) & \frac{\partial z}{\partial w}(\mathbb{u}_0) \\ \frac{\partial x}{\partial v}(\mathbb{u}_0) & \frac{\partial x}{\partial w}(\mathbb{u}_0) \end{vmatrix}$$

$$+ \frac{\partial y}{\partial u}(\mathbb{u}_0) \frac{\partial h}{\partial y}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial x}{\partial v}(\mathbb{u}_0) & \frac{\partial x}{\partial w}(\mathbb{u}_0) \\ \frac{\partial y}{\partial v}(\mathbb{u}_0) & \frac{\partial y}{\partial w}(\mathbb{u}_0) \end{vmatrix} + \frac{\partial z}{\partial u}(\mathbb{u}_0) \frac{\partial f}{\partial z}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial y}{\partial v}(\mathbb{u}_0) & \frac{\partial y}{\partial w}(\mathbb{u}_0) \\ \frac{\partial z}{\partial v}(\mathbb{u}_0) & \frac{\partial z}{\partial w}(\mathbb{u}_0) \end{vmatrix}$$

$$\left. + \frac{\partial z}{\partial u}(\mathbb{u}_0) \frac{\partial g}{\partial z}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial z}{\partial v}(\mathbb{u}_0) & \frac{\partial z}{\partial w}(\mathbb{u}_0) \\ \frac{\partial x}{\partial v}(\mathbb{u}_0) & \frac{\partial x}{\partial w}(\mathbb{u}_0) \end{vmatrix} + \frac{\partial z}{\partial u}(\mathbb{u}_0) \frac{\partial h}{\partial z}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial x}{\partial v}(\mathbb{u}_0) & \frac{\partial x}{\partial w}(\mathbb{u}_0) \\ \frac{\partial y}{\partial v}(\mathbb{u}_0) & \frac{\partial y}{\partial w}(\mathbb{u}_0) \end{vmatrix} \right\}$$

[By the same token as in (3.3)] (3.5)

$$\int_{w_0}^{w_0+d_3} \int_{u_0}^{u_0+d_1} \mathbb{f}(\mathbb{x}(u, v_0, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial w}(u, v_0, w) \times \frac{\partial \mathbb{x}}{\partial u}(u, v_0, w) \right) dw\, du$$

$$= d_1 d_3 \mathbb{f}(\mathbb{x}_0) \cdot \left(\frac{\partial \mathbb{x}}{\partial w}(\mathbb{u}_0) \times \frac{\partial \mathbb{x}}{\partial u}(\mathbb{u}_0) \right) \quad \text{[Proposition 2.1]} \qquad (3.6)$$

$$\int_{w_0}^{w_0+d_3} \int_{u_0}^{u_0+d_1} \mathbb{f}(\mathbb{x}(u, v_0 + d_2, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial w}(u, v_0 + d_2, w) \right.$$

$$\left. \times \frac{\partial \mathbb{x}}{\partial u}(u, v_0 + d_2, w) \right) dw\, du$$

$$
\begin{aligned}
&= d_1 d_3 \mathbb{f}(\mathbb{x}_0) \cdot \left(\frac{\partial \mathbb{x}}{\partial w}(\mathbb{u}_0) \times \frac{\partial \mathbb{x}}{\partial u}(\mathbb{u}_0) \right) + d_1 d_2 d_3 \Bigg\{ \mathbb{f}(\mathbb{x}_0) \cdot \left(\frac{\partial \mathbb{x}}{\partial w}(\mathbb{u}_0) \right. \\
&\quad \left. \times \frac{\partial^2 \mathbb{x}}{\partial u\, \partial v}(\mathbb{u}_0) \right) + \mathbb{f}(\mathbb{x}_0) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(\mathbb{u}_0) \times \frac{\partial^2 \mathbb{x}}{\partial v\, \partial w}(\mathbb{u}_0) \right) \\
&\quad + \frac{\partial x}{\partial v}(\mathbb{u}_0) \frac{\partial f}{\partial x}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial y}{\partial w}(\mathbb{u}_0) & \frac{\partial y}{\partial u}(\mathbb{u}_0) \\ \frac{\partial z}{\partial w}(\mathbb{u}_0) & \frac{\partial z}{\partial u}(\mathbb{u}_0) \end{vmatrix} + \frac{\partial x}{\partial v}(\mathbb{u}_0) \frac{\partial g}{\partial x}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial z}{\partial w}(\mathbb{u}_0) & \frac{\partial z}{\partial u}(\mathbb{u}_0) \\ \frac{\partial x}{\partial w}(\mathbb{u}_0) & \frac{\partial x}{\partial u}(\mathbb{u}_0) \end{vmatrix} \\
&\quad + \frac{\partial x}{\partial v}(\mathbb{u}_0) \frac{\partial h}{\partial x}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial x}{\partial w}(\mathbb{u}_0) & \frac{\partial x}{\partial u}(\mathbb{u}_0) \\ \frac{\partial y}{\partial w}(\mathbb{u}_0) & \frac{\partial y}{\partial u}(\mathbb{u}_0) \end{vmatrix} + \frac{\partial y}{\partial v}(\mathbb{u}_0) \frac{\partial f}{\partial y}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial y}{\partial w}(\mathbb{u}_0) & \frac{\partial y}{\partial u}(\mathbb{u}_0) \\ \frac{\partial z}{\partial w}(\mathbb{u}_0) & \frac{\partial z}{\partial u}(\mathbb{u}_0) \end{vmatrix} \\
&\quad + \frac{\partial y}{\partial v}(\mathbb{u}_0) \frac{\partial g}{\partial y}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial z}{\partial w}(\mathbb{u}_0) & \frac{\partial z}{\partial u}(\mathbb{u}_0) \\ \frac{\partial x}{\partial w}(\mathbb{u}_0) & \frac{\partial x}{\partial u}(\mathbb{u}_0) \end{vmatrix} + \frac{\partial y}{\partial v}(\mathbb{u}_0) \frac{\partial h}{\partial y}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial x}{\partial w}(\mathbb{u}_0) & \frac{\partial x}{\partial u}(\mathbb{u}_0) \\ \frac{\partial y}{\partial w}(\mathbb{u}_0) & \frac{\partial y}{\partial u}(\mathbb{u}_0) \end{vmatrix} \\
&\quad + \frac{\partial z}{\partial v}(\mathbb{u}_0) \frac{\partial f}{\partial z}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial y}{\partial w}(\mathbb{u}_0) & \frac{\partial y}{\partial u}(\mathbb{u}_0) \\ \frac{\partial z}{\partial w}(\mathbb{u}_0) & \frac{\partial z}{\partial u}(\mathbb{u}_0) \end{vmatrix} + \frac{\partial z}{\partial v}(\mathbb{u}_0) \frac{\partial g}{\partial z}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial z}{\partial w}(\mathbb{u}_0) & \frac{\partial z}{\partial u}(\mathbb{u}_0) \\ \frac{\partial x}{\partial w}(\mathbb{u}_0) & \frac{\partial x}{\partial u}(\mathbb{u}_0) \end{vmatrix} \\
&\quad + \frac{\partial z}{\partial v}(\mathbb{u}_0) \frac{\partial h}{\partial z}(\mathbb{x}_0) \begin{vmatrix} \frac{\partial x}{\partial w}(\mathbb{u}_0) & \frac{\partial x}{\partial u}(\mathbb{u}_0) \\ \frac{\partial y}{\partial w}(\mathbb{u}_0) & \frac{\partial y}{\partial u}(\mathbb{u}_0) \end{vmatrix} \Bigg\} \quad \text{[By the same token as in (3.3)]}
\end{aligned}
\tag{3.7}
$$

It is well known in elementary linear algebra (Laplace's expansion theorem, cf. Ihara (1982), theorem 4.6) that

$$
\begin{aligned}
&\frac{\partial x}{\partial u}(\mathbb{u}_0) \begin{vmatrix} \frac{\partial y}{\partial v}(\mathbb{u}_0) & \frac{\partial y}{\partial w}(\mathbb{u}_0) \\ \frac{\partial z}{\partial v}(\mathbb{u}_0) & \frac{\partial z}{\partial w}(\mathbb{u}_0) \end{vmatrix} + \frac{\partial x}{\partial v}(\mathbb{u}_0) \begin{vmatrix} \frac{\partial y}{\partial w}(\mathbb{u}_0) & \frac{\partial y}{\partial u}(\mathbb{u}_0) \\ \frac{\partial z}{\partial w}(\mathbb{u}_0) & \frac{\partial z}{\partial u}(\mathbb{u}_0) \end{vmatrix} \\
&\quad + \frac{\partial x}{\partial w}(\mathbb{u}_0) \begin{vmatrix} \frac{\partial y}{\partial u}(\mathbb{u}_0) & \frac{\partial y}{\partial v}(\mathbb{u}_0) \\ \frac{\partial z}{\partial u}(\mathbb{u}_0) & \frac{\partial z}{\partial v}(\mathbb{u}_0) \end{vmatrix} = \left[\frac{\partial \mathbb{x}}{\partial u}(\mathbb{u}_0), \frac{\partial \mathbb{x}}{\partial v}(\mathbb{u}_0), \frac{\partial \mathbb{x}}{\partial w}(\mathbb{u}_0) \right]
\end{aligned}
\tag{3.8}
$$

$$\frac{\partial y}{\partial u}(\mathbb{w}_0)\begin{vmatrix}\frac{\partial z}{\partial v}(\mathbb{w}_0) & \frac{\partial z}{\partial w}(\mathbb{w}_0)\\ \frac{\partial x}{\partial v}(\mathbb{w}_0) & \frac{\partial z}{\partial w}(\mathbb{w}_0)\end{vmatrix} + \frac{\partial y}{\partial v}(\mathbb{w}_0)\begin{vmatrix}\frac{\partial z}{\partial w}(\mathbb{w}_0) & \frac{\partial z}{\partial u}(\mathbb{w}_0)\\ \frac{\partial x}{\partial w}(\mathbb{w}_0) & \frac{\partial x}{\partial u}(\mathbb{w}_0)\end{vmatrix}$$
$$+ \frac{\partial y}{\partial w}(\mathbb{w}_0)\begin{vmatrix}\frac{\partial z}{\partial u}(\mathbb{w}_0) & \frac{\partial z}{\partial v}(\mathbb{w}_0)\\ \frac{\partial x}{\partial u}(\mathbb{w}_0) & \frac{\partial x}{\partial v}(\mathbb{w}_0)\end{vmatrix} = \left[\frac{\partial \mathbb{x}}{\partial u}(\mathbb{w}_0), \frac{\partial \mathbb{x}}{\partial v}(\mathbb{w}_0), \frac{\partial \mathbb{x}}{\partial w}(\mathbb{w}_0)\right] \tag{3.9}$$

$$\frac{\partial z}{\partial u}(\mathbb{w}_0)\begin{vmatrix}\frac{\partial x}{\partial v}(\mathbb{w}_0) & \frac{\partial x}{\partial w}(\mathbb{w}_0)\\ \frac{\partial y}{\partial v}(\mathbb{w}_0) & \frac{\partial y}{\partial w}(\mathbb{w}_0)\end{vmatrix} + \frac{\partial z}{\partial v}(\mathbb{w}_0)\begin{vmatrix}\frac{\partial x}{\partial w}(\mathbb{w}_0) & \frac{\partial x}{\partial u}(\mathbb{w}_0)\\ \frac{\partial y}{\partial w}(\mathbb{w}_0) & \frac{\partial y}{\partial u}(\mathbb{w}_0)\end{vmatrix}$$
$$+ \frac{\partial z}{\partial w}(\mathbb{w}_0)\begin{vmatrix}\frac{\partial x}{\partial u}(\mathbb{w}_0) & \frac{\partial x}{\partial v}(\mathbb{w}_0)\\ \frac{\partial y}{\partial u}(\mathbb{w}_0) & \frac{\partial y}{\partial v}(\mathbb{w}_0)\end{vmatrix} = \left[\frac{\partial \mathbb{x}}{\partial u}(\mathbb{w}_0), \frac{\partial \mathbb{x}}{\partial v}(\mathbb{w}_0), \frac{\partial \mathbb{x}}{\partial w}(\mathbb{w}_0)\right] \tag{3.10}$$

$$\frac{\partial y}{\partial u}(\mathbb{w}_0)\begin{vmatrix}\frac{\partial y}{\partial v}(\mathbb{w}_0) & \frac{\partial y}{\partial w}(\mathbb{w}_0)\\ \frac{\partial z}{\partial v}(\mathbb{w}_0) & \frac{\partial z}{\partial w}(\mathbb{w}_0)\end{vmatrix} + \frac{\partial y}{\partial v}(\mathbb{w}_0)\begin{vmatrix}\frac{\partial y}{\partial w}(\mathbb{w}_0) & \frac{\partial y}{\partial u}(\mathbb{w}_0)\\ \frac{\partial z}{\partial w}(\mathbb{w}_0) & \frac{\partial z}{\partial u}(\mathbb{w}_0)\end{vmatrix}$$
$$+ \frac{\partial y}{\partial w}(\mathbb{w}_0)\begin{vmatrix}\frac{\partial y}{\partial u}(\mathbb{w}_0) & \frac{\partial y}{\partial v}(\mathbb{w}_0)\\ \frac{\partial z}{\partial u}(\mathbb{w}_0) & \frac{\partial z}{\partial v}(\mathbb{w}_0)\end{vmatrix} = 0 \tag{3.11}$$

$$\frac{\partial z}{\partial u}(\mathbb{w}_0)\begin{vmatrix}\frac{\partial y}{\partial v}(\mathbb{w}_0) & \frac{\partial y}{\partial w}(\mathbb{w}_0)\\ \frac{\partial z}{\partial v}(\mathbb{w}_0) & \frac{\partial z}{\partial w}(\mathbb{w}_0)\end{vmatrix} + \frac{\partial z}{\partial v}(\mathbb{w}_0)\begin{vmatrix}\frac{\partial y}{\partial w}(\mathbb{w}_0) & \frac{\partial y}{\partial u}(\mathbb{w}_0)\\ \frac{\partial z}{\partial w}(\mathbb{w}_0) & \frac{\partial z}{\partial u}(\mathbb{w}_0)\end{vmatrix}$$
$$+ \frac{\partial z}{\partial w}(\mathbb{w}_0)\begin{vmatrix}\frac{\partial y}{\partial u}(\mathbb{w}_0) & \frac{\partial y}{\partial v}(\mathbb{w}_0)\\ \frac{\partial z}{\partial u}(\mathbb{w}_0) & \frac{\partial z}{\partial v}(\mathbb{w}_0)\end{vmatrix} = 0 \tag{3.12}$$

$$
\begin{aligned}
&\frac{\partial z}{\partial u}(\mathbb{u}_0)\begin{vmatrix}\frac{\partial z}{\partial v}(\mathbb{u}_0) & \frac{\partial z}{\partial w}(\mathbb{u}_0)\\ \frac{\partial x}{\partial v}(\mathbb{u}_0) & \frac{\partial x}{\partial w}(\mathbb{u}_0)\end{vmatrix}+\frac{\partial z}{\partial v}(\mathbb{u}_0)\begin{vmatrix}\frac{\partial z}{\partial w}(\mathbb{u}_0) & \frac{\partial z}{\partial u}(\mathbb{u}_0)\\ \frac{\partial x}{\partial w}(\mathbb{u}_0) & \frac{\partial x}{\partial u}(\mathbb{u}_0)\end{vmatrix}\\
&\quad+\frac{\partial z}{\partial w}(\mathbb{u}_0)\begin{vmatrix}\frac{\partial z}{\partial u}(\mathbb{u}_0) & \frac{\partial z}{\partial v}(\mathbb{u}_0)\\ \frac{\partial x}{\partial u}(\mathbb{u}_0) & \frac{\partial x}{\partial v}(\mathbb{u}_0)\end{vmatrix}=0 \qquad (3.13)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial x}{\partial u}(\mathbb{u}_0)\begin{vmatrix}\frac{\partial z}{\partial v}(\mathbb{u}_0) & \frac{\partial z}{\partial w}(\mathbb{u}_0)\\ \frac{\partial x}{\partial v}(\mathbb{u}_0) & \frac{\partial x}{\partial w}(\mathbb{u}_0)\end{vmatrix}+\frac{\partial x}{\partial v}(\mathbb{u}_0)\begin{vmatrix}\frac{\partial z}{\partial w}(\mathbb{u}_0) & \frac{\partial z}{\partial u}(\mathbb{u}_0)\\ \frac{\partial x}{\partial w}(\mathbb{u}_0) & \frac{\partial x}{\partial u}(\mathbb{u}_0)\end{vmatrix}\\
&\quad+\frac{\partial x}{\partial w}(\mathbb{u}_0)\begin{vmatrix}\frac{\partial z}{\partial u}(\mathbb{u}_0) & \frac{\partial z}{\partial v}(\mathbb{u}_0)\\ \frac{\partial x}{\partial u}(\mathbb{u}_0) & \frac{\partial x}{\partial v}(\mathbb{u}_0)\end{vmatrix}=0 \qquad (3.14)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial x}{\partial u}(\mathbb{u}_0)\begin{vmatrix}\frac{\partial x}{\partial v}(\mathbb{u}_0) & \frac{\partial x}{\partial w}(\mathbb{u}_0)\\ \frac{\partial y}{\partial v}(\mathbb{u}_0) & \frac{\partial y}{\partial w}(\mathbb{u}_0)\end{vmatrix}+\frac{\partial x}{\partial v}(\mathbb{u}_0)\begin{vmatrix}\frac{\partial x}{\partial w}(\mathbb{u}_0) & \frac{\partial x}{\partial u}(\mathbb{u}_0)\\ \frac{\partial y}{\partial w}(\mathbb{u}_0) & \frac{\partial y}{\partial u}(\mathbb{u}_0)\end{vmatrix}\\
&\quad+\frac{\partial x}{\partial w}(\mathbb{u}_0)\begin{vmatrix}\frac{\partial x}{\partial u}(\mathbb{u}_0) & \frac{\partial x}{\partial v}(\mathbb{u}_0)\\ \frac{\partial y}{\partial u}(\mathbb{u}_0) & \frac{\partial y}{\partial v}(\mathbb{u}_0)\end{vmatrix}=0 \qquad (3.15)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial y}{\partial u}(\mathbb{u}_0)\begin{vmatrix}\frac{\partial x}{\partial v}(\mathbb{u}_0) & \frac{\partial x}{\partial w}(\mathbb{u}_0)\\ \frac{\partial y}{\partial v}(\mathbb{u}_0) & \frac{\partial y}{\partial w}(\mathbb{u}_0)\end{vmatrix}+\frac{\partial y}{\partial v}(\mathbb{u}_0)\begin{vmatrix}\frac{\partial x}{\partial w}(\mathbb{u}_0) & \frac{\partial x}{\partial u}(\mathbb{u}_0)\\ \frac{\partial y}{\partial w}(\mathbb{u}_0) & \frac{\partial y}{\partial u}(\mathbb{u}_0)\end{vmatrix}\\
&\quad+\frac{\partial y}{\partial w}(\mathbb{u}_0)\begin{vmatrix}\frac{\partial x}{\partial u}(\mathbb{u}_0) & \frac{\partial x}{\partial v}(\mathbb{u}_0)\\ \frac{\partial y}{\partial u}(\mathbb{u}_0) & \frac{\partial y}{\partial v}(\mathbb{u}_0)\end{vmatrix}=0 \qquad (3.16)
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&(3.3) + (3.5) + (3.7) - (3.2) - (3.4) - (3.6) \\
&\quad = (\operatorname{div} \mathbb{f})(\mathbb{x}_0) \left[\frac{\partial \mathbb{x}}{\partial u}(\mathbb{u}_0), \frac{\partial \mathbb{x}}{\partial v}(\mathbb{u}_0), \frac{\partial \mathbb{x}}{\partial w}(\mathbb{u}_0) \right] \\
&\quad = \int_{u_0}^{u_0+d_1} \int_{v_0}^{v_0+d_2} \int_{w_0}^{w_0+d_3} (\operatorname{div} \mathbb{f}) \left[\frac{\partial \mathbb{x}}{\partial u}, \frac{\partial \mathbb{x}}{\partial v}, \frac{\partial \mathbb{x}}{\partial w} \right] du\, dv\, dw \quad \text{[Proposition 2.1]}
\end{aligned}
\tag{3.17}
$$

This completes the proof. □

Now we would like to derive the usual local version of divergence theorem from Theorem 3.1 step by step. Let us begin with

Lemma 3.2. *We have*

$$
\begin{aligned}
&\int_{u_0}^{u_1} \int_{v_0}^{v_0+d_2} \int_{w_0}^{w_0+d_3} (\operatorname{div} \mathbb{f}) \left[\frac{\partial \mathbb{x}}{\partial u}, \frac{\partial \mathbb{x}}{\partial v}, \frac{\partial \mathbb{x}}{\partial w} \right] du\, dv\, dw \\
&\quad = \int_{u_0}^{u_1} \int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u, v, w_0 + d_3)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u, v, w_0 + d_3) \right. \\
&\qquad \left. \times \frac{\partial \mathbb{x}}{\partial v}(u, v, w_0 + d_3) \right) du\, dv + \int_{v_0}^{v_0+d_2} \int_{w_0}^{w_0+d_3} \mathbb{f}(\mathbb{x}(u_1, v, w)) \cdot \\
&\qquad \left(\frac{\partial \mathbb{x}}{\partial u}(u_1, v, w) \times \frac{\partial \mathbb{x}}{\partial v}(u_1, v, w) \right) dv\, dw + \int_{w_0}^{w_0+d_3} \int_{u_0}^{u_1} \\
&\qquad \mathbb{f}(\mathbb{x}(u, v_0 + d_2, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial w}(u, v_0 + d_2, w) \times \frac{\partial \mathbb{x}}{\partial u}(u, v_0 + d_2, w) \right) dw\, du \\
&\qquad - \int_{u_0}^{u_1} \int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u, v, w_0)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u, v, w_0) \times \frac{\partial \mathbb{x}}{\partial v}(u, v, w_0) \right) du\, dv \\
&\qquad - \int_{v_0}^{v_0+d_2} \int_{w_0}^{w_0+d_3} \mathbb{f}(\mathbb{x}(u_0, v, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u_0, v, w) \times \frac{\partial \mathbb{x}}{\partial v}(u_0, v, w) \right) dv\, dw \\
&\qquad - \int_{w_0}^{w_0+d_3} \int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u, v_0, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial w}(u, v_0, w) \times \frac{\partial \mathbb{x}}{\partial u}(u, v_0, w) \right) dw\, du
\end{aligned}
\tag{3.18}
$$

Proof: Let us define a function $f : [u_0, u_1] \to \mathbb{R}$. For each $u_2 \in [u_0, u_1]$ we decree that

$$
\begin{aligned}
f(u_2) = {} & \int_{u_0}^{u_2}\int_{v_0}^{v_0+d_2}\int_{w_0}^{w_0+d_3} (\mathrm{div}\,\mathbb{f})\left[\frac{\partial \mathbb{x}}{\partial u}, \frac{\partial \mathbb{x}}{\partial v}, \frac{\partial \mathbb{x}}{\partial w}\right] du\,dv\,dw - \int_{u_0}^{u_2}\int_{v_0}^{v_0+d_2} \\
& \mathbb{f}(\mathbb{x}(u, v, w_0 + d_3)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u, v, w_0 + d_3) \times \frac{\partial \mathbb{x}}{\partial v}(u, v, w_0 + d_3)\right) du\,dv \\
& - \int_{v_0}^{v_0+d_2}\int_{w_0}^{w_0+d_3} \mathbb{f}(\mathbb{x}(u_2, v, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u_2, v, w) \times \frac{\partial \mathbb{x}}{\partial v}(u_2, v, w)\right) dv\,dw \\
& - \int_{w_0}^{w_0+d_3}\int_{u_0}^{u_2} \mathbb{f}(\mathbb{x}(u, v_0, d_2, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial w}(u, v_0 + d_2, w)\right. \\
& \left. \times \frac{\partial \mathbb{x}}{\partial u}(u, v_0 + d_2, w)\right) dw\,du + \int_{u_0}^{u_2}\int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u, v, w_0)) \cdot \\
& \left(\frac{\partial \mathbb{x}}{\partial u}(u, v, w_0) \times \frac{\partial \mathbb{x}}{\partial v}(u, v, w_0)\right) du\,dv + \int_{v_0}^{v_0+d_2}\int_{w_0}^{w_0+d_3} \\
& \mathbb{f}(\mathbb{x}(u_0, v, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u_0, v, w) \times \frac{\partial \mathbb{x}}{\partial v}(u_0, v, w)\right) dv\,dw \\
& + \int_{w_0}^{w_0+d_3}\int_{u_0}^{u_2} \mathbb{f}(\mathbb{x}(u, v_0, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial w}(u, v_0, w) \times \frac{\partial \mathbb{x}}{\partial u}(u, v_0, w)\right) dw\,du
\end{aligned}
\tag{3.19}
$$

For each $d \in \mathrm{D}$ we have

$$
\begin{aligned}
& f(u_2 + d) - f(d) \\
& = \int_{u_0}^{u_2+d}\int_{v_0}^{v_0+d_2}\int_{w_0}^{w_0+d_3} (\mathrm{div}\,\mathbb{f})\left[\frac{\partial \mathbb{x}}{\partial u}, \frac{\partial \mathbb{x}}{\partial v}, \frac{\partial \mathbb{x}}{\partial w}\right] du\,dv\,dw - \int_{u_0}^{u_2+d}\int_{v_0}^{v_0+d_2} \\
& \mathbb{f}(\mathbb{x}(u, v, w_0 + d_3)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u, v, w_0 + d_3) \times \frac{\partial \mathbb{x}}{\partial v}(u, v, w_0 + d_3)\right) du\,dv \\
& - \int_{v_0}^{v_0+d_2}\int_{w_0}^{w_0+d_3} \mathbb{f}(\mathbb{x}(u_2 + d, v, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u_2 + d, v, w)\right. \\
& \left. \times \frac{\partial \mathbb{x}}{\partial v}(u_2 + d, v, w)\right) dv\,dw - \int_{w_0}^{w_0+d_3}\int_{u_0}^{u_2+d} \mathbb{f}(\mathbb{x}(u, v_0 + d_2, w)) \cdot \\
& \left(\frac{\partial \mathbb{x}}{\partial w}(u, v_0 + d_2, w) \times \frac{\partial \mathbb{x}}{\partial u}(u, v_0 + d_2, w)\right) dw\,du + \int_{u_0}^{u_2+d}\int_{v_0}^{v_0+d_2} \\
& \mathbb{f}(\mathbb{x}(u, v, w_0)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u, v, w_0) \times \frac{\partial \mathbb{x}}{\partial v}(u, v, w_0)\right) du\,dv + \int_{v_0}^{v_0+d_2}\int_{w_0}^{w_0+d_3}
\end{aligned}
$$

$$\mathbb{f}(\mathbb{x}(u_0, v, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u_0, v, w) \times \frac{\partial \mathbb{x}}{\partial v}(u_0, v, w)\right) dv\, dw + \int_{w_0}^{w_0+d_3} \int_{u_0}^{u_2}$$

$$\mathbb{f}(\mathbb{x}(u, v_0, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial w}(u, v_0, w) \times \frac{\partial \mathbb{x}}{\partial u}(u, v_0, w)\right) dw\, du$$

$$- \int_{u_0}^{u_2} \int_{v_0}^{v_0+d_2} \int_{w_0}^{w_0+d_3} (\operatorname{div} \mathbb{f}) \left[\frac{\partial \mathbb{x}}{\partial u}, \frac{\partial \mathbb{x}}{\partial v}, \frac{\partial \mathbb{x}}{\partial w}\right] du\, dv\, dw + \int_{u_0}^{u_2} \int_{v_0}^{v_0+d_2}$$

$$\mathbb{f}(\mathbb{x}(u, v, w_0 + d_3)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u, v, w_0 + d_3) \times \frac{\partial \mathbb{x}}{\partial v}(u, v, w_0 + d_3)\right) du\, dv$$

$$+ \int_{v_0}^{v_0+d_2} \int_{w_0}^{w_0+d_3} \mathbb{f}(\mathbb{x}(u_2, v, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u_2, v, w) \times \frac{\partial \mathbb{x}}{\partial v}(u_2, v, w)\right) dv\, dw$$

$$+ \int_{w_0}^{w_0+d_3} \int_{u_0}^{u_2} \mathbb{f}(\mathbb{x}(u, v_0 + d_2, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial w}(u, v_0 + d_2, w)\right.$$

$$\left.\times \frac{\partial \mathbb{x}}{\partial u}(u, v_0 + d_2, w)\right) dw\, du - \int_{u_0}^{u_2} \int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u, v, w_0)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u, v, w_0)\right.$$

$$\left.\times \frac{\partial \mathbb{x}}{\partial v}(u, v, w_0)\right) du\, dv - \int_{v_0}^{v_0+d_2} \int_{w_0}^{w_0+d_3} \mathbb{f}(\mathbb{x}(u_0, v, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u_0, v, w)\right.$$

$$\left.\times \frac{\partial \mathbb{x}}{\partial v}(u_0, v, w)\right) dv\, dw - \int_{w_0}^{w_0+d_3} \int_{u_0}^{u_2} \mathbb{f}(\mathbb{x}(u, v_0, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial w}(u, v_0, w)\right.$$

$$\left.\times \frac{\partial \mathbb{x}}{\partial u}(u, v_0, w)\right) dw\, du$$

$$= \int_{u_2}^{u_2+d} \int_{v_0}^{v_0+d_2} \int_{w_0}^{w_0+d_3} (\operatorname{div} \mathbb{f}) \left[\frac{\partial \mathbb{x}}{\partial u}, \frac{\partial \mathbb{x}}{\partial v}, \frac{\partial \mathbb{x}}{\partial w}\right] du\, dv\, dw - \int_{u_2}^{u_2+d} \int_{v_0}^{v_0+d_2}$$

$$\mathbb{f}(\mathbb{x}(u, v, w_0 + d_3)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u, v, w_0 + d_3) \times \frac{\partial \mathbb{x}}{\partial v}(u, v, w_0 + d_3)\right) du\, dv$$

$$- \int_{v_0}^{v_0+d_2} \int_{w_0}^{w_0+d_3} \mathbb{f}(\mathbb{x}(u_2 + d, v, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u_2 + d, v, w)\right.$$

$$\left.\times \frac{\partial \mathbb{x}}{\partial v}(u_2 + d, v, w)\right) dv\, dw - \int_{w_0}^{w_0+d_3} \int_{u_2}^{u_2+d} \mathbb{f}(\mathbb{x}(u, v_0 + d_2, w)) \cdot$$

$$\left(\frac{\partial \mathbb{x}}{\partial w}(u, v_0 + d_2, w) \times \frac{\partial \mathbb{x}}{\partial u}(u, v_0 + d_2, w)\right) dw\, du + \int_{u_2}^{u_2+d} \int_{v_0}^{v_0+d_2}$$

$$\mathbb{f}(\mathbb{x}(u, v, w_0)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u, v, w_0) \times \frac{\partial \mathbb{x}}{\partial v}(u, v, w_0)\right) du\, dv + \int_{v_0}^{v_0+d_2} \int_{w_0}^{w_0+d_3}$$

$$\mathbb{f}(\mathbb{x}(u_2, v, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u_2, v, w) \times \frac{\partial \mathbb{x}}{\partial v}(u_2, v, w) \right) dv\, dw + \int_{w_0}^{w_0+d_3} \int_{u_2}^{u_2+d}$$

$$\mathbb{f}(\mathbb{x}(u, v_0, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial w}(u, v_0, w) \times \frac{\partial \mathbb{x}}{\partial u}(u, v_0, w) \right) dw\, du = 0$$

$$\text{[Theorem 3.1]} \tag{3.20}$$

This means that $f' = 0$ on $[u_0, u_1]$, so that f is constant on $[u_0, u_1]$. Since $f(u_0) = 0$ trivially, we have $f = 0$ on $[u_0, u_1]$. In particular, $f(u_1) = 0$, which is tantamount to (3.18). □

Next we will show that

Lemma 3.3. *We have*

$$\int_{u_0}^{u_1} \int_{v_0}^{v_1} \int_{w_0}^{w_0+d_3} (\mathrm{div}\, \mathbb{f}) \left[\frac{\partial \mathbb{x}}{\partial u}, \frac{\partial \mathbb{x}}{\partial v}, \frac{\partial \mathbb{x}}{\partial w} \right] du\, dv\, dw$$

$$= \int_{u_0}^{u_1} \int_{v_0}^{v_1} \mathbb{f}(\mathbb{x}(u, v, w_0 + d_3)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u, v, w_0 + d_3) \right.$$

$$\left. \times \frac{\partial \mathbb{x}}{\partial v}(u, v, w_0 + d_3) \right) du\, dv + \int_{v_0}^{v_1} \int_{w_0}^{w_0+d_3} \mathbb{f}(\mathbb{x}(u_1, v, w)) \cdot$$

$$\left(\frac{\partial \mathbb{x}}{\partial u}(u_1, v, w) \times \frac{\partial \mathbb{x}}{\partial v}(u_1, v, w) \right) dv\, dw + \int_{w_0}^{w_0+d_3} \int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u, v_1, w)) \cdot$$

$$\left(\frac{\partial \mathbb{x}}{\partial w}(u, v_1, w) \times \frac{\partial \mathbb{x}}{\partial u}(u, v_1, w) \right) dw\, du - \int_{u_0}^{u_1} \int_{v_0}^{v_1} \mathbb{f}(\mathbb{x}(u, v, w_0)) \cdot$$

$$\left(\frac{\partial \mathbb{x}}{\partial u}(u, v, w_0) \times \frac{\partial \mathbb{x}}{\partial v}(u, v, w_0) \right) du\, dv - \int_{v_0}^{v_1} \int_{w_0}^{w_0+d_3} \mathbb{f}(\mathbb{x}(u_0, v, w)) \cdot$$

$$\left(\frac{\partial \mathbb{x}}{\partial u}(u_0, v, w) \cdot \times \frac{\partial \mathbb{x}}{\partial v}(u_0, v, w) \right) dv\, dw - \int_{w_0}^{w_0+d_3} \int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u, v_0, w)) \cdot$$

$$\left(\frac{\partial \mathbb{x}}{\partial w}(u, v_0, w) \times \frac{\partial \mathbb{x}}{\partial u}(u, v_0, w) \right) dw\, du \tag{3.21}$$

Proof: Let us define a function $g : [v_0, v_1] \to \mathbb{R}$. For any $v_2 \in [v_0, v_1]$ we decree that

$$g(v_2) = \int_{u_0}^{u_1} \int_{v_0}^{v_2} \int_{w_0}^{w_0+d_3} (\mathrm{div}\, \mathbb{f}) \left[\frac{\partial \mathbb{x}}{\partial u}, \frac{\partial \mathbb{x}}{\partial v}, \frac{\partial \mathbb{x}}{\partial w} \right] du\, dv\, dw - \int_{u_0}^{u_1} \int_{v_0}^{v_2}$$

$$\mathbb{f}(\mathbb{x}(u, v, w_0 + d_3)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u, v, w_0 + d_3) \times \frac{\partial \mathbb{x}}{\partial v}(u, v, w_0 + d_3) \right) du\, dv$$

$$-\int_{v_0}^{v_2}\int_{w_0}^{w_0+d_3} \mathbb{f}(\mathbb{x}(u_1,v,w))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u_1,v,w)\times\frac{\partial\mathbb{x}}{\partial v}(u_1,v,w)\right)dv\,dw$$
$$-\int_{w_0}^{w_0+d_3}\int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u,v_2,w))\cdot\left(\frac{\partial\mathbb{x}}{\partial w}(u,v_2,w)\times\frac{\partial\mathbb{x}}{\partial u}(u,v_2,w)\right)dw\,du$$
$$+\int_{u_0}^{u_1}\int_{v_0}^{v_2} \mathbb{f}(\mathbb{x}(u,v,w_0))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u,v,w_0)\times\frac{\partial\mathbb{x}}{\partial v}(u,v,w_0)\right)du\,dv$$
$$+\int_{v_0}^{v_2}\int_{w_0}^{w_0+d_3} \mathbb{f}(\mathbb{x}(u_0,v,w))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u_0,v,w)\times\frac{\partial\mathbb{x}}{\partial v}(u_0,v,w)\right)dv\,dw$$
$$+\int_{w_0}^{w_0+d_3}\int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u,v_0,w))\cdot\left(\frac{\partial\mathbb{x}}{\partial w}(u,v_0,w)\times\frac{\partial\mathbb{x}}{\partial u}(u,v_0,w)\right)dw\,du \tag{3.22}$$

For any $d \in \mathrm{D}$ we have

$$g(v_2+d)-g(v_2)$$
$$=\int_{u_0}^{u_1}\int_{v_0}^{v_2+d}\int_{w_0}^{w_0+d_3}(\operatorname{div}\mathbb{f})\left[\frac{\partial\mathbb{x}}{\partial u},\frac{\partial\mathbb{x}}{\partial v},\frac{\partial\mathbb{x}}{\partial w}\right]du\,dv\,dw-\int_{u_0}^{u_1}\int_{v_0}^{v_2+d}$$
$$\mathbb{f}(\mathbb{x}(u,v,w_0+d_3))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u,v,w_0+d_3)\times\frac{\partial\mathbb{x}}{\partial v}(u,v,w_0+d_3)\right)du\,dv$$
$$-\int_{v_0}^{v_2+d}\int_{w_0}^{w_0+d_3} \mathbb{f}(\mathbb{x}(u_1,v,w))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u_1,v,w)\times\frac{\partial\mathbb{x}}{\partial v}(u_1,v,w)\right)dv\,dw$$
$$-\int_{w_0}^{w_0+d_3}\int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u,v_2+d,w))\cdot\left(\frac{\partial\mathbb{x}}{\partial w}(u,v_2+d,w)\right.$$
$$\left.\times\frac{\partial\mathbb{x}}{\partial u}(u,v_2+d,w)\right)dw\,du+\int_{u_0}^{u_1}\int_{v_2}^{v_2+d} \mathbb{f}(\mathbb{x}(u,v,w_0))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u,v,w_0)\right.$$
$$\left.\times\frac{\partial\mathbb{x}}{\partial v}(u,v,w_0)\right)du\,dv+\int_{v_0}^{v_2+d}\int_{w_0}^{w_0+d_3} \mathbb{f}(\mathbb{x}(u_0,v,w))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u_0,v,w)\right.$$
$$\left.\times\frac{\partial\mathbb{x}}{\partial v}(u_0,v,w)\right)dv\,dw+\int_{w_0}^{w_0+d_3}\int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u,v_0,w))\cdot\left(\frac{\partial\mathbb{x}}{\partial w}(u,v_0,w)\right.$$
$$\left.\times\frac{\partial\mathbb{x}}{\partial u}(u,v_0,w)\right)dw\,du-\int_{u_0}^{u_1}\int_{v_0}^{v_2}\int_{w_0}^{w_0+d_3}(\operatorname{div}\mathbb{f})\left[\frac{\partial\mathbb{x}}{\partial u},\frac{\partial\mathbb{x}}{\partial v},\frac{\partial\mathbb{x}}{\partial w}\right]du\,dv\,dw$$

$$+\int_{u_0}^{u_1}\int_{v_0}^{v_2}\mathbb{f}(\mathbb{x}(u,v,w_0+d_3))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u,v,w_0+d_3)\right.$$
$$\left.\times\frac{\partial\mathbb{x}}{\partial v}(u,v,w_0+d_3)\right)du\,dv+\int_{v_0}^{v_2}\int_{w_0}^{w_0+d_3}\mathbb{f}(\mathbb{x}(u_1,v,w))\cdot$$
$$\left(\frac{\partial\mathbb{x}}{\partial u}(u_1,v,w)\times\frac{\partial\mathbb{x}}{\partial v}(u_1,v,w)\right)dv\,dw+\int_{w_0}^{w_0+d_3}\int_{u_0}^{u_1}\mathbb{f}(\mathbb{x}(u,v_2,w))\cdot$$
$$\left(\frac{\partial\mathbb{x}}{\partial w}(u,v_2,w)\times\frac{\partial\mathbb{x}}{\partial u}(u,v_2,w)\right)dw\,du-\int_{u_0}^{u_1}\int_{v_0}^{v_2}\mathbb{f}(\mathbb{x}(u,v,w_0))\cdot$$
$$\left(\frac{\partial\mathbb{x}}{\partial u}(u,v,w_0)\times\frac{\partial\mathbb{x}}{\partial v}(u,v,w_0)\right)du\,dv-\int_{v_0}^{v_2}\int_{w_0}^{w_0+d_3}\mathbb{f}(\mathbb{x}(u_0,v,w))\cdot$$
$$\left(\frac{\partial\mathbb{x}}{\partial u}(u_0,v,w)\times\frac{\partial\mathbb{x}}{\partial v}(u_0,v,w)\right)dv\,dw-\int_{w_0}^{w_0+d_3}\int_{u_0}^{u_1}\mathbb{f}(\mathbb{x}(u,v_0,w))\cdot$$
$$\left(\frac{\partial\mathbb{x}}{\partial w}(u,v_0,w)\times\frac{\partial\mathbb{x}}{\partial u}(u,v_0,w)\right)dw\,du$$
$$=\int_{u_0}^{u_1}\int_{v_2}^{v_2+d}\int_{w_0}^{w_0+d_3}(\mathrm{div}\,\mathbb{f})\left[\frac{\partial\mathbb{x}}{\partial u},\frac{\partial\mathbb{x}}{\partial v},\frac{\partial\mathbb{x}}{\partial w}\right]du\,dv\,dw-\int_{u_0}^{u_1}\int_{v_2}^{v_2+d}$$
$$\mathbb{f}(\mathbb{x}(u,v,w_0+d_3))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u,v,w_0+d_3)\times\frac{\partial\mathbb{x}}{\partial v}(u,v,w_0+d_3)\right)du\,dv$$
$$-\int_{v_2}^{v_2+d}\int_{w_0}^{w_0+d_3}\mathbb{f}(\mathbb{x}(u_1,v,w))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u_1,v,w)\times\frac{\partial\mathbb{x}}{\partial v}(u_1,v,w)\right)dv\,dw$$
$$-\int_{w_0}^{w_0+d_3}\int_{u_0}^{u_1}\mathbb{f}(\mathbb{x}(u,v_2+d,w))\cdot\left(\frac{\partial\mathbb{x}}{\partial w}(u,v_2+d,w)\right.$$
$$\left.\times\frac{\partial\mathbb{x}}{\partial u}(u,v_2+d,w)\right)dw\,du+\int_{u_0}^{u_1}\int_{v_2}^{v_2+d}\mathbb{f}(\mathbb{x}(u,v,w_0))\cdot$$
$$\left(\frac{\partial\mathbb{x}}{\partial u}(u,v,w_0)\times\frac{\partial\mathbb{x}}{\partial v}(u,v,w_0)\right)du\,dv+\int_{v_2}^{v_2+d}\int_{w_0}^{w_0+d_3}\mathbb{f}(\mathbb{x}(u_0,v,w))\cdot$$
$$\left(\frac{\partial\mathbb{x}}{\partial u}(u_0,v,w)\times\frac{\partial\mathbb{x}}{\partial v}(u_0,v,w)\right)dv\,dw+\int_{w_0}^{w_0+d_3}\int_{u_0}^{u_1}\mathbb{f}(\mathbb{x}(u,v_2,w))\cdot$$
$$\left(\frac{\partial\mathbb{x}}{\partial w}(u,v_2,w)\times\frac{\partial\mathbb{x}}{\partial u}(u,v_2,w)\right)dw\,du=0\quad\text{[Lemma 3.2]}\tag{3.23}$$

This means that $g'=0$ on $[v_0,v_1]$, so that g is constant on $[v_0,v_1]$. Since $g(v_0)=0$ trivially, we have $g=0$ on $[v_0,v_1]$. In particular, $g(v_1)=0$, which is tantamount to (3.21). □

Finally we have

Theorem 3.4 (*Divergence Theorem*)**.**

$$\begin{aligned}
&\int_{u_0}^{u_1}\int_{v_0}^{v_1}\int_{w_0}^{w_1}(\operatorname{div}\mathbb{f})\left[\frac{\partial\mathbb{x}}{\partial u},\frac{\partial\mathbb{x}}{\partial v},\frac{\partial\mathbb{x}}{\partial w}\right]du\,dv\,dw\\
&\quad=\int_{u_0}^{u_1}\int_{v_0}^{v_1}\mathbb{f}(\mathbb{x}(u,v,w_1))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u,v,w_1)\times\frac{\partial\mathbb{x}}{\partial v}(u,v,w_1)\right)du\,dv\\
&\qquad+\int_{v_0}^{v_1}\int_{w_0}^{w_1}\mathbb{f}(\mathbb{x}(u_1,v,w))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u_1,v,w)\times\frac{\partial\mathbb{x}}{\partial v}(u_1,v,w)\right)dv\,dw\\
&\qquad+\int_{w_0}^{w_1}\int_{u_0}^{u_1}\mathbb{f}(\mathbb{x}(u,v_1,w))\cdot\left(\frac{\partial\mathbb{x}}{\partial w}(u,v_1,w)\times\frac{\partial\mathbb{x}}{\partial u}(u,v_1,w)\right)dw\,du\\
&\qquad-\int_{u_0}^{u_1}\int_{v_0}^{v_1}\mathbb{f}(\mathbb{x}(u,v,w_0))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u,v,w_0)\times\frac{\partial\mathbb{x}}{\partial v}(u,v,w_0)\right)du\,dv\\
&\qquad-\int_{v_0}^{v_1}\int_{w_0}^{w_1}\mathbb{f}(\mathbb{x}(u_0,v,w))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u_0,v,w)\times\frac{\partial\mathbb{x}}{\partial v}(u_0,v,w)\right)dv\,dw\\
&\qquad-\int_{w_0}^{w_1}\int_{u_0}^{u_1}\mathbb{f}(\mathbb{x}(u,v_0,w))\cdot\left(\frac{\partial\mathbb{x}}{\partial w}(u,v_0,w)\times\frac{\partial\mathbb{x}}{\partial u}(u,v_0,w)\right)dw\,du \qquad (3.24)
\end{aligned}$$

Proof: Let us define a function $h:[v_0,v_1]\to\mathbb{R}$. For any $w_2\in[w_0,w_1]$ we decree that

$$\begin{aligned}
h(w_2)={}&\int_{u_0}^{u_1}\int_{v_0}^{v_1}\int_{w_0}^{w_2}(\operatorname{div}\mathbb{f})\left[\frac{\partial\mathbb{x}}{\partial u},\frac{\partial\mathbb{x}}{\partial v},\frac{\partial\mathbb{x}}{\partial w}\right]du\,dv\,dw-\int_{u_0}^{u_1}\int_{v_0}^{v_1}\mathbb{f}(\mathbb{x}(u,v,w_2))\cdot\\
&\left(\frac{\partial\mathbb{x}}{\partial u}(u,v,w_2)\times\frac{\partial\mathbb{x}}{\partial v}(u,v,w_2)\right)du\,dv-\int_{v_0}^{v_1}\int_{w_0}^{w_2}\mathbb{f}(\mathbb{x}(u_1,v,w))\cdot\\
&\left(\frac{\partial\mathbb{x}}{\partial u}(u_1,v,w)\times\frac{\partial\mathbb{x}}{\partial v}(u_1,v,w)\right)dv\,dw-\int_{w_0}^{w_2}\int_{u_0}^{u_1}\mathbb{f}(\mathbb{x}(u,v_1,w))\cdot\\
&\left(\frac{\partial\mathbb{x}}{\partial w}(u,v_1,w)\times\frac{\partial\mathbb{x}}{\partial u}(u,v_1,w)\right)dw\,du+\int_{u_0}^{u_1}\int_{v_0}^{v_1}\mathbb{f}(\mathbb{x}(u,v,w_0))\cdot\\
&\left(\frac{\partial\mathbb{x}}{\partial u}(u,v,w_0)\times\frac{\partial\mathbb{x}}{\partial v}(u,v,w_0)\right)du\,dv+\int_{v_0}^{v_1}\int_{w_0}^{w_2}\mathbb{f}(\mathbb{x}(u_0,v,w))\cdot\\
&\left(\frac{\partial\mathbb{x}}{\partial u}(u_0,v,w)\times\frac{\partial\mathbb{x}}{\partial v}(u_0,v,w)\right)dv\,dw+\int_{w_0}^{w_2}\int_{u_0}^{u_1}\mathbb{f}(\mathbb{x}(u,v_0,w))\cdot\\
&\left(\frac{\partial\mathbb{x}}{\partial w}(u,v_0,w)\times\frac{\partial\mathbb{x}}{\partial u}(u,v_0,w)\right)dw\,du \qquad (3.25)
\end{aligned}$$

For each $d \in \mathrm{D}$ we have

$$
\begin{aligned}
&h(w_2+d)-h(w_2)\\
&=\int_{u_0}^{u_1}\int_{v_0}^{v_1}\int_{w_0}^{w_2+d}(\mathrm{div}\,\mathbb{f})\left[\frac{\partial\mathbb{x}}{\partial u},\frac{\partial\mathbb{x}}{\partial v},\frac{\partial\mathbb{x}}{\partial w}\right]du\,dv\,dw-\int_{u_0}^{u_1}\int_{v_0}^{v_1}\\
&\quad\mathbb{f}(\mathbb{x}(u,v,w_2+d))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u,v,w_2+d)\times\frac{\partial\mathbb{x}}{\partial v}(u,v,w_2+d)\right)du\,dv\\
&\quad-\int_{v_0}^{v_1}\int_{w_0}^{w_2+d}\mathbb{f}(\mathbb{x}(u_1,v,w))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u_1,v,w)\times\frac{\partial\mathbb{x}}{\partial v}(u_1,v,w)\right)dv\,dw\\
&\quad-\int_{w_0}^{w_2+d}\int_{u_0}^{u_1}\mathbb{f}(\mathbb{x}(u,v_1,w))\cdot\left(\frac{\partial\mathbb{x}}{\partial w}(u,v_1,w)\times\frac{\partial\mathbb{x}}{\partial u}(u,v_1,w)\right)dw\,du\\
&\quad+\int_{u_0}^{u_1}\int_{v_0}^{v_1}\mathbb{f}(\mathbb{x}(u,v,w_0))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u,v,w_0)\times\frac{\partial\mathbb{x}}{\partial v}(u,v,w_0)\right)du\,dv\\
&\quad+\int_{v_0}^{v_1}\int_{w_0}^{w_2+d}\mathbb{f}(\mathbb{x}(u_0,v,w))\cdot\left(\frac{\partial\mathbb{x}}{\partial u}(u_0,v,w)\times\frac{\partial\mathbb{x}}{\partial v}(u_0,v,w)\right)dv\,dw\\
&\quad+\int_{w_0}^{w_2+d}\int_{u_0}^{u_1}\mathbb{f}(\mathbb{x}(u,v_0,w))\cdot\left(\frac{\partial\mathbb{x}}{\partial w}(u,v_0,w)\times\frac{\partial\mathbb{x}}{\partial u}(u,v_0,w)\right)dw\,du\\
&\quad-\int_{u_0}^{u_1}\int_{v_0}^{v_1}\int_{w_0}^{w_2}(\mathrm{div}\,\mathbb{f})\left[\frac{\partial\mathbb{x}}{\partial u},\frac{\partial\mathbb{x}}{\partial v},\frac{\partial\mathbb{x}}{\partial w}\right]du\,dv\,dw+\int_{u_0}^{u_1}\int_{v_0}^{v_1}\mathbb{f}(\mathbb{x}(u,v,w_2))\cdot\\
&\quad\left(\frac{\partial\mathbb{x}}{\partial u}(u,v,w_2)\times\frac{\partial\mathbb{x}}{\partial v}(u,v,w_2)\right)du\,dv+\int_{v_0}^{v_1}\int_{w_0}^{w_2}\mathbb{f}(\mathbb{x}(u_1,v,w))\cdot\\
&\quad\left(\frac{\partial\mathbb{x}}{\partial u}(u_1,v,w)\times\frac{\partial\mathbb{x}}{\partial v}(u_1,v,w)\right)dv\,dw+\int_{w_0}^{w_2}\int_{u_0}^{u_1}\mathbb{f}(\mathbb{x}(u,v_1,w))\cdot\\
&\quad\left(\frac{\partial\mathbb{x}}{\partial w}(u,v_1,w)\times\frac{\partial\mathbb{x}}{\partial u}(u,v_1,w)\right)dw\,du-\int_{u_0}^{u_1}\int_{v_0}^{v_1}\mathbb{f}(\mathbb{x}(u,v,w_0))\cdot\\
&\quad\left(\frac{\partial\mathbb{x}}{\partial u}(u,v,w_0)\times\frac{\partial\mathbb{x}}{\partial v}(u,v,w_0)\right)du\,dv-\int_{v_0}^{v_1}\int_{w_0}^{w_2}\mathbb{f}(\mathbb{x}(u_0,v,w))\cdot\\
&\quad\left(\frac{\partial\mathbb{x}}{\partial u}(u_0,v,w)\times\frac{\partial\mathbb{x}}{\partial v}(u_0,v,w)\right)dv\,dw-\int_{w_0}^{w_2}\int_{u_0}^{u_1}\mathbb{f}(\mathbb{x}(u,v_0,w))\cdot\\
&\quad\left(\frac{\partial\mathbb{x}}{\partial w}(u,v_0,w)\times\frac{\partial\mathbb{x}}{\partial u}(u,v_0,w)\right)dw\,du
\end{aligned}
$$

$$= \int_{u_0}^{u_1}\int_{v_0}^{v_1}\int_{w_2}^{w_2+d} (\mathrm{div}\, \mathbb{f}) \left[\frac{\partial \mathbb{x}}{\partial u}, \frac{\partial \mathbb{x}}{\partial v}, \frac{\partial \mathbb{x}}{\partial w}\right] du\, dv\, dw - \int_{u_0}^{u_1}\int_{v_0}^{v_1}$$
$$\mathbb{f}(\mathbb{x}(u, v, w_2 + d)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u, v, w_2 + d) \times \frac{\partial \mathbb{x}}{\partial v}(u, v, w_2 + d)\right) du\, dv$$
$$-\int_{v_0}^{v_1}\int_{w_2}^{w_2+d} \mathbb{f}(\mathbb{x}(u_1, v, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u_1, v, w) \times \frac{\partial \mathbb{x}}{\partial v}(u_1, v, w)\right) dv\, dw$$
$$-\int_{w_2}^{w_2+d}\int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u, v_1, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial w}(u, v_1, w) \times \frac{\partial \mathbb{x}}{\partial u}(u, v_1, w)\right) dw\, du$$
$$+\int_{u_0}^{u_1}\int_{v_0}^{v_1} \mathbb{f}(\mathbb{x}(u, v, w_2)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u, v, w_2) \times \frac{\partial \mathbb{x}}{\partial v}(u, v, w_2)\right) du\, dv$$
$$+\int_{v_0}^{v_1}\int_{w_2}^{w_2+d} \mathbb{f}(\mathbb{x}(u_0, v, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u_0, v, w) \times \frac{\partial \mathbb{x}}{\partial v}(u_0, v, w)\right) dv\, dw$$
$$+\int_{w_2}^{w_2+d}\int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u, v_0, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial w}(u, v_0, w) \times \frac{\partial \mathbb{x}}{\partial u}(u, v_0, w)\right) dw\, du = 0$$
$$[\text{Lemma 3.3}] \tag{3.26}$$

This means that $h' = 0$ on $[w_0, w_1]$, so that h is constant on $[w_0, w_1]$. Since $h(w_0) = 0$ trivially, we have $h = 0$ on $[w_0, w_1]$. In particular, $h(w_1) = 0$, which is tantamount to (3.24). □

4. ROTATION

Let $\mathbb{f}$ be a vector field as in the previous section. Let $u_0, v_0, u_1, v_1 \in \mathbb{R}$ with $u_0 \leq u_1$ and $v_0 \leq v_1$. We assume that we have a function $[u_0, u_1] \times [v_0, v_1] \to \mathbb{R}^3$ assigning $\mathbb{x}(\mathbb{u}) = \mathbb{x}(u, v) = (x(u, v), y(u, v), z(u, v)) \in \mathbb{R}^3$ to each $\mathbb{u} = (u, v) \in [u_0, u_1] \times [v_0, v_1]$. Let $d_1, d_2 \in \mathrm{D}$. Let $\mathbb{u}_0 = (u_0, v_0)$ and $\mathbb{x}_0 = (x_0, y_0, z_0) = \mathbb{x}(\mathbb{u}_0)$. Then we have

Theorem 4.1 (*Infinitesimal Stokes's Theorem*)**.**

$$\int_{u_0}^{u_0+d_1}\int_{v_0}^{v_0+d_2} (\mathrm{rot}\, \mathbb{f}) \cdot \left(\frac{\partial \mathbb{x}}{\partial u} \times \frac{\partial \mathbb{x}}{\partial v}\right) du\, dv$$
$$= \int_{u_0}^{u_0+d_1} \mathbb{f}(\mathbb{x}(u, v_0)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u, v_0)\right) du + \int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u_0 + d_1, v)) \cdot$$
$$\frac{\partial \mathbb{x}}{\partial v}(u_0 + d_1, v)\, dv - \int_{u_0}^{u_0+d_1} \mathbb{f}(\mathbb{x}(u, v_0 + d_2)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_0 + d_2)\, du$$
$$- \int_{v_0}^{v_0+d_2} \times\ \mathbb{f}(\mathbb{x}(u_0, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_0, v)\, dv, \tag{4.1}$$

where rot $\mathbb{f}$ *stands for* $(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})$.

Proof: We have

$$\int_{u_0}^{u_0+d_1} \mathbb{f}(\mathbb{x}(u, v_0)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_0)\, du = d_1 \mathbb{f}(\mathbb{x}(\mathbb{u}_0)) \cdot \frac{\partial \mathbb{x}}{\partial u}(\mathbb{u}_0) \quad \text{[Proposition 2.1]} \tag{4.2}$$

$$\begin{aligned}
&\int_{u_0}^{u_0+d_1} \mathbb{f}(\mathbb{x}(u, v_0 + d2)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_0 + d_2)\, du \\
&= d_1 \mathbb{f}(\mathbb{x}(u_0, v_0 + d_2)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u_0, v_0 + d_2) \quad \text{[Proposition 2.1]} \\
&= d_1 \mathbb{f}\left(\mathbb{x}_0 + d_2 \frac{\partial \mathbb{x}}{\partial v}(\mathbb{u}_0)\right) \cdot \left\{\frac{\partial \mathbb{x}}{\partial u}(\mathbb{u}_0) + d_2 \frac{\partial^2 \mathbb{x}}{\partial u\, \partial v}(\mathbb{u}_0)\right\} \\
&= d_1 \left\{\mathbb{f}(\mathbb{x}_0) + d_2 \frac{\partial x}{\partial v}(\mathbb{u}_0) \frac{\partial \mathbb{f}}{\partial x}(\mathbb{x}_0) + d_2 \frac{\partial y}{\partial v}(\mathbb{u}_0) \frac{\partial \mathbb{f}}{\partial y}(\mathbb{x}_0)\right. \\
&\quad \left. + d_2 \frac{\partial z}{\partial v}(\mathbb{u}_0) \frac{\partial \mathbb{f}}{\partial z}(\mathbb{x}_0)\right\} \cdot \left\{\frac{\partial \mathbb{x}}{\partial u}(\mathbb{u}_0) + d_2 \frac{\partial^2 \mathbb{x}}{\partial u\, \partial v}(\mathbb{u}_0)\right\} \\
&= d_1 \left\{\mathbb{f}(\mathbb{x}_0) \cdot \frac{\partial \mathbb{x}}{\partial u}(\mathbb{u}_0) + d_2 \mathbb{f}(\mathbb{x}_0) \cdot \frac{\partial^2 \mathbb{x}}{\partial u\, \partial v}(\mathbb{u}_0) + d_2 \frac{\partial x}{\partial u}(\mathbb{u}_0) \frac{\partial x}{\partial v}(\mathbb{u}_0) \frac{\partial f}{\partial x}(\mathbb{x}_0)\right. \\
&\quad + d_2 \frac{\partial y}{\partial u}(\mathbb{u}_0) \frac{\partial y}{\partial v}(\mathbb{u}_0) \frac{\partial g}{\partial y}(\mathbb{x}_0) + d_2 \frac{\partial z}{\partial u}(\mathbb{u}_0) \frac{\partial z}{\partial v}(\mathbb{u}_0) \frac{\partial h}{\partial z}(\mathbb{x}_0) \\
&\quad + d_2 \frac{\partial y}{\partial u}(\mathbb{u}_0) \frac{\partial x}{\partial v}(\mathbb{u}_0) \frac{\partial g}{\partial x}(\mathbb{x}_0) + d_2 \frac{\partial z}{\partial u}(\mathbb{u}_0) \frac{\partial x}{\partial v}(\mathbb{u}_0) \frac{\partial h}{\partial x}(\mathbb{x}_0) \\
&\quad + d_2 \frac{\partial x}{\partial u}(\mathbb{u}_0) \frac{\partial y}{\partial v}(\mathbb{u}_0) \frac{\partial f}{\partial y}(\mathbb{x}_0) + d_2 \frac{\partial z}{\partial u}(\mathbb{u}_0) \frac{\partial y}{\partial v}(\mathbb{u}_0) \frac{\partial h}{\partial y}(\mathbb{x}_0) \\
&\quad \left. + d_2 \frac{\partial x}{\partial u}(\mathbb{u}_0) \frac{\partial z}{\partial v}(\mathbb{u}_0) \frac{\partial f}{\partial z}(\mathbb{x}_0) + d_2 \frac{\partial y}{\partial u}(\mathbb{u}_0) \frac{\partial z}{\partial v}(\mathbb{u}_0) \frac{\partial g}{\partial z}(\mathbb{x}_0)\right\}
\end{aligned} \tag{4.3}$$

$$\int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u_0, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_0, v)\, dv = d_2 \mathbb{f}(\mathbb{x}(\mathbb{u}_0)) \cdot \frac{\partial \mathbb{x}}{\partial v}(\mathbb{u}_0) \quad \text{[Proposition 2.1]} \tag{4.4}$$

$$\begin{aligned}
&\int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u_0 + d_1, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_0 + d_1, v)\, dv \\
&= d_2 \left\{\mathbb{f}(\mathbb{x}_0) \cdot \frac{\partial \mathbb{x}}{\partial v}(\mathbb{u}_0) + d_1 \mathbb{f}(\mathbb{x}_0) \cdot \frac{\partial^2 \mathbb{x}}{\partial u\, \partial v}(\mathbb{u}_0) + d_1 \frac{\partial x}{\partial u}(\mathbb{u}_0) \frac{\partial x}{\partial v}(\mathbb{u}_0) \frac{\partial f}{\partial x}(\mathbb{x}_0)\right. \\
&\quad + d_1 \frac{\partial y}{\partial u}(\mathbb{u}_0) \frac{\partial y}{\partial v}(\mathbb{u}_0) \frac{\partial g}{\partial y}(\mathbb{x}_0) + d_1 \frac{\partial z}{\partial u}(\mathbb{u}_0) \frac{\partial z}{\partial v}(\mathbb{u}_0) \frac{\partial h}{\partial z}(\mathbb{x}_0)
\end{aligned}$$

$$+ d_1 \frac{\partial x}{\partial u}(\mathbb{u}_0)\frac{\partial y}{\partial v}(\mathbb{u}_0)\frac{\partial g}{\partial x}(\mathbb{x}_0) + d_1 \frac{\partial x}{\partial u}(\mathbb{u}_0)\frac{\partial z}{\partial v}(\mathbb{u}_0)\frac{\partial h}{\partial x}(\mathbb{x}_0)$$
$$+ d_1 \frac{\partial u}{\partial u}(\mathbb{u}_0)\frac{\partial x}{\partial v}(\mathbb{u}_0)\frac{\partial f}{\partial y}(\mathbb{x}_0) + d_1 \frac{\partial y}{\partial u}(\mathbb{u}_0)\frac{\partial z}{\partial v}(\mathbb{u}_0)\frac{\partial h}{\partial y}(\mathbb{x}_0)$$
$$\left. + d_1 \frac{\partial z}{\partial u}(\mathbb{u}_0)\frac{\partial x}{\partial v}(\mathbb{u}_0)\frac{\partial f}{\partial z}(\mathbb{x}_0) + d_1 \frac{\partial z}{\partial u}(\mathbb{u}_0)\frac{\partial y}{\partial v}(\mathbb{u}_0)\frac{\partial g}{\partial z}(\mathbb{x}_0)\right\} \text{[By the same token as in (4.3)]} \quad (4.5)$$

$$(4.2) + (4.5) - (4.3) - (4.4)$$
$$= d_1 d_2 \left\{ \left(\frac{\partial h}{\partial y}(\mathbb{u}_0) - \frac{\partial g}{\partial z}(\mathbb{u}_0) \right) \begin{vmatrix} \frac{\partial y}{\partial u}(\mathbb{u}_0) & \frac{\partial y}{\partial v}(\mathbb{u}_0) \\ \frac{\partial z}{\partial u}(\mathbb{u}_0) & \frac{\partial z}{\partial v}(\mathbb{u}_0) \end{vmatrix} \right.$$
$$+ \left(\frac{\partial f}{\partial z}(\mathbb{u}_0) - \frac{\partial h}{\partial x}(\mathbb{u}_0) \right) \begin{vmatrix} \frac{\partial z}{\partial u}(\mathbb{u}_0) & \frac{\partial z}{\partial v}(\mathbb{u}_0) \\ \frac{\partial x}{\partial u}(\mathbb{u}_0) & \frac{\partial x}{\partial v}(\mathbb{u}_0) \end{vmatrix}$$
$$\left. + \left(\frac{\partial g}{\partial x}(\mathbb{u}_0) - \frac{\partial f}{\partial y}(\mathbb{u}_0) \right) \begin{vmatrix} \frac{\partial x}{\partial u}(\mathbb{u}_0) & \frac{\partial x}{\partial v}(\mathbb{u}_0) \\ \frac{\partial y}{\partial u}(\mathbb{u}_0) & \frac{\partial y}{\partial v}(\mathbb{u}_0) \end{vmatrix} \right\}$$
$$= \int_{u_0}^{u_0+d_1} \int_{v_0}^{v_0+d_2} (\text{rot } \mathbb{f}) \cdot \left(\frac{\partial \mathbb{x}}{\partial u} \times \frac{\partial \mathbb{x}}{\partial v} \right) du\, dv \quad \text{[Proposition 2.1]} \quad (4.6)$$

This completes the proof. □

Now we would like to derive the usual local version of Stokes's theorem from Theorem 4.1. We need the following lemma.

Lemma 4.2. *We have*

$$\int_{u_0}^{u_1} \int_{v_0}^{v_0+d_2} (\text{rot } \mathbb{f}) \cdot \left(\frac{\partial \mathbb{x}}{\partial u} \times \frac{\partial \mathbb{x}}{\partial v} \right) du\, dv$$
$$= \int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u, v_0)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_0)\, du + \int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u_1, v)) \cdot$$
$$\frac{\partial \mathbb{x}}{\partial v}(u_1, v)\, dv - \int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u, v_0 + d_2)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_0 + d_2)\, du$$
$$- \int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u_0, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_0, v)\, dv \quad (4.7)$$

Proof: Let us define a function $f : [u_0, u_1] \to \mathbb{R}$. For each $u_2 \in [u_0, u_1]$ we decree that

$$f(u_2) = \int_{u_0}^{u_2}\int_{v_0}^{v_0+d_2} (\mathrm{rot}\ \mathbb{f}) \cdot \left(\frac{\partial \mathbb{x}}{\partial u} \times \frac{\partial \mathbb{x}}{\partial v}\right) du\, dv - \int_{u_0}^{u_2} \mathbb{f}(\mathbb{x}(u, v_0)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_0)\, du$$
$$- \int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u_2, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_2, v)\, dv + \int_{u_0}^{u_2} \mathbb{f}(\mathbb{x}(u, v_0 + d_2)) \cdot$$
$$\frac{\partial \mathbb{x}}{\partial u}(u, v_0 + d_2)\, du + \int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u_0, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_0, v)\, dv[1pt] \tag{4.8}$$

For each $d \in D$ we have

$$f(u_2 + d) - f(u_2)$$
$$= \int_{u_0}^{u_2+d}\int_{v_0}^{v_0+d_2} (\mathrm{rot}\ \mathbb{f}) \cdot \left(\frac{\partial \mathbb{x}}{\partial u} \times \frac{\partial \mathbb{x}}{\partial v}\right) du\, dv - \int_{u_0}^{u_2+d} \mathbb{f}(\mathbb{x}(u, v_0)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_0)\, du$$
$$- \int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u_2 + d, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_2 + d, v)\, dv + \int_{u_0}^{u_2+d} \mathbb{f}(\mathbb{x}(u, v_0 + d_2)) \cdot$$
$$\frac{\partial \mathbb{x}}{\partial u}(u, v_0 + d_2)\, du + \int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u_0, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_0, v)\, dv$$
$$- \int_{u_0}^{u_2}\int_{v_0}^{v_0+d_2} (\mathrm{rot}\ \mathbb{f}) \cdot \left(\frac{\partial \mathbb{x}}{\partial u} \times \frac{\partial \mathbb{x}}{\partial v}\right) du\, dv + \int_{u_0}^{u_2} \mathbb{f}(\mathbb{x}(u, v_0)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_0)\, du$$
$$+ \int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u_2, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_2, v)\, dv - \int_{u_0}^{u_2} \mathbb{f}(\mathbb{x}(u, v_0 + d_2)) \cdot$$
$$\frac{\partial \mathbb{x}}{\partial u}(u, v_0 + d_2)\, du - \int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u_0, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_0, v)\, dv$$
$$= \int_{u_2}^{u_2+d}\int_{v_0}^{v_0+d_2} (\mathrm{rot}\ \mathbb{f}) \cdot \left(\frac{\partial \mathbb{x}}{\partial u} \times \frac{\partial \mathbb{x}}{\partial v}\right) du\, dv - \int_{u_2}^{u_2+d} \mathbb{f}(\mathbb{x}(u, v_0)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_0)\, du \cdot$$
$$- \int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u_2 + d, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_2 + d, v)\, dv + \int_{u_2}^{u_2+d} \mathbb{f}(\mathbb{x}(u, v_0 + d_2)) \cdot$$
$$\frac{\partial \mathbb{x}}{\partial u}(u, v_0 + d_2)\, du + \int_{v_0}^{v_0+d_2} \mathbb{f}(\mathbb{x}(u_2, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_2, v)\, dv = 0$$
$$[\text{Theorem 4.1}] \tag{4.9}$$

This means that $f' = 0$ on $[u_0, u_1]$, so that f is constant on $[u_0, u_1]$. Since $f(u_0) = 0$ trivially, we have $f = 0$ on $[u_0, u_1]$. In particular, $f(u_1) = 0$, which is tantamout to (4.7). □

Theorem 4.3 (*Stokes's Theorem*). *We have*

$$\int_{u_0}^{u_1}\int_{v_0}^{v_1} (\mathrm{rot}\,\mathbb{f}) \cdot \left(\frac{\partial \mathbb{x}}{\partial u} \times \frac{\partial \mathbb{x}}{\partial v}\right) du\,dv$$
$$= \int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u, v_0)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_0)\,du + \int_{v_0}^{v_1} \mathbb{f}(\mathbb{x}(u_1, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_1, v)\,dv$$
$$- \int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u, v_1)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_1)\,du - \int_{v_0}^{v_1} \mathbb{f}(\mathbb{x}(u_0, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_0, v)\,dv \quad (4.10)$$

Proof: Let us define a function $g : [v_0, v_1] \to \mathbb{R}$. For any $v_2 \in [v_0, v_1]$ we decree that

$$g(v_2) = \int_{u_0}^{u_1}\int_{v_0}^{v_2} (\mathrm{rot}\,\mathbb{f}) \cdot \left(\frac{\partial \mathbb{x}}{\partial u} \times \frac{\partial \mathbb{x}}{\partial v}\right) du\,dv - \int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u, v_0)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_0)\,du$$
$$- \int_{v_0}^{v_2} \mathbb{f}(\mathbb{x}(u_1, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_1, v)\,dv + \int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u, v_2)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_2)\,du$$
$$+ \int_{v_0}^{v_2} \mathbb{f}(\mathbb{x}(u_0, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_0, v)\,dv \quad (4.11)$$

For any $d \in \mathrm{D}$ we have

$$g(v_2 + d) - g(v_2)$$
$$= \int_{u_0}^{u_1}\int_{v_0}^{v_2+d} (\mathrm{rot}\,\mathbb{f}) \cdot \left(\frac{\partial \mathbb{x}}{\partial u} \times \frac{\partial \mathbb{x}}{\partial v}\right) du\,dv - \int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u, v_0)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_0)\,du$$
$$- \int_{v_0}^{v_2+d} \mathbb{f}(\mathbb{x}(u_1, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_1, v)\,dv + \int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u, v_2 + d)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_2 + d)\,du$$
$$+ \int_{v_0}^{v_2+d} \mathbb{f}(\mathbb{x}(u_0, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_0, v)\,dv - \int_{u_0}^{u_1}\int_{v_0}^{v_2} (\mathrm{rot}\,\mathbb{f}) \cdot \left(\frac{\partial \mathbb{x}}{\partial u} \times \frac{\partial \mathbb{x}}{\partial v}\right) du\,dv$$
$$+ \int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u, v_0)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_0)\,du + \int_{v_0}^{v_2} \mathbb{f}(\mathbb{x}(u_1, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_1, v)\,dv$$

$$-\int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u, v_2)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_2)\, du - \int_{v_0}^{v_2} \mathbb{f}(\mathbb{x}(u_0, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_0, v)\, dv$$

$$= \int_{u_0}^{u_1}\int_{v_2}^{v_2+d} (\mathrm{rot}\, \mathbb{f}) \cdot \left(\frac{\partial \mathbb{x}}{\partial u} \times \frac{\partial \mathbb{x}}{\partial v}\right) du\, dv - \int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u, v_2)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_2)\, du$$
$$-\int_{v_2}^{v_2+d} \mathbb{f}(\mathbb{x}(u_1, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_1, v)\, dv + \int_{u_0}^{u_1} \mathbb{f}(\mathbb{x}(u, v_2+d)) \cdot \frac{\partial \mathbb{x}}{\partial u}(u, v_2+d)\, du$$
$$+\int_{v_2}^{v_2+d} \mathbb{f}(\mathbb{x}(u_0, v)) \cdot \frac{\partial \mathbb{x}}{\partial v}(u_0, v)\, dv = 0 \quad \text{[Lemma 4.2]} \tag{4.12}$$

This means that $g' = 0$ on $[v_0, v_1]$, so that g is constant on $[v_0, v_1]$. Since $g(v_0) = 0$ trivially, we have $g = 0$ on $[v_0, v_1]$. In particular, $g(v_1) = 0$, which is tantamount to (4.10). □

5. div ∘ rot = 0

In this section we will give a highly geometric proof of div ∘ rot $= 0$.

Theorem 5.1. *We have*

$$\mathrm{div}(\mathrm{rot}\, \mathbb{f}) = 0 \tag{5.1}$$

Proof: On the one hand we have

$$\int_{u_0}^{u_0+d_1}\int_{v_0}^{v_0+d_2}\int_{w_0}^{w_0+d_3} \mathrm{div}(\mathrm{rot}\, \mathbb{f}) \left[\frac{\partial \mathbb{x}}{\partial u}, \frac{\partial \mathbb{x}}{\partial v}, \frac{\partial \mathbb{x}}{\partial w}\right] du\, dv\, dw$$
$$= \int_{u_0}^{u_0+d_1}\int_{v_0}^{v_0+d_2} (\mathrm{rot}\, \mathbb{f}(u, v, w_0+d_3)) \cdot \left(\frac{\partial \mathbb{x}}{\partial u}(u, v, w_0+d_3)\right.$$
$$\left. \times \frac{\partial \mathbb{x}}{\partial v}(u, v, w_0+d_3)\right) du\, dv + \int_{v_0}^{v_0+d_1}\int_{w_0}^{w_0+d_2} (\mathrm{rot}\, \mathbb{f}(u_0+d_1, v, w)) \cdot$$
$$\left(\frac{\partial \mathbb{x}}{\partial v}(u_0+d_1, v, w) \times \frac{\partial \mathbb{x}}{\partial w}(u_0+d_1, v, w)\right) dv\, dw$$
$$+\int_{v_0}^{w_0+d_3}\int_{u_0}^{u_0+d_1} (\mathrm{rot}\, \mathbb{f}(u, v_0+d_2, w)) \cdot \left(\frac{\partial \mathbb{x}}{\partial w}(u, v_0+d_2, w)\right.$$
$$\left. \times \frac{\partial \mathbb{x}}{\partial u}(u, v_0+d_2, w)\right) du\, dw - \int_{u_0}^{u_0+d_1}\int_{v_0}^{v_0+d_2} (\mathrm{rot}\, \mathbb{f}(u, v, w_0)) \cdot$$
$$\left(\frac{\partial \mathbb{x}}{\partial u}(u, v, w_0) \times \frac{\partial \mathbb{x}}{\partial v}(u, v, w_0)\right) du\, dv - \int_{v_0}^{v_0+d_2}\int_{w_0}^{w_0+d_3} (\mathrm{rot}\, \mathbb{f}(u_0, v, w)) \cdot$$
$$\left(\frac{\partial \mathbb{x}}{\partial v}(u_0, v, w) \times \frac{\partial \mathbb{x}}{\partial w}(u_0, v, w)\right) dv\, dw - \int_{w_0}^{w_0+d_3}\int_{u_0}^{u_0+d_1} (\mathrm{rot}\, \mathbb{f}(u, v_0, w)) \cdot$$

$$
\begin{aligned}
&\left(\frac{\partial \mathbb{X}}{\partial w}(u, v_0, w) \times \frac{\partial \mathbb{X}}{\partial u}(u, v_0, w)\right) du\, dw \quad \text{[Theorem 3.1]}\\
&= \left\{\int_{u_0}^{u_0+d_1} \mathbb{f}(u, v_0, w_0+d_3) \cdot \frac{\partial \mathbb{X}}{\partial u}(u, v_0, w_0+d_3)\, du\right.\\
&\quad + \int_{v_0}^{v_0+d_2} \mathbb{f}(u_0+d_1, v, w_0+d_3) \cdot \frac{\partial \mathbb{X}}{\partial v}(u_0+d_1, v, w_0+d_3)\, dv\\
&\quad - \int_{u_0}^{u_0+d_1} \mathbb{f}(u, v_0+d_2, w_0+d_3) \cdot \frac{\partial \mathbb{X}}{\partial u}(u, v_0+d_2, w_0+d_3)\, du\\
&\quad \left.- \int_{v_0}^{v_0+d_2} \mathbb{f}(u_0, v, w_0+d_3) \cdot \frac{\partial \mathbb{X}}{\partial v}(u_0, v, w_0+d_3)\, dv\right\}\\
&\quad + \left\{\int_{v_0}^{v_0+d_2} \mathbb{f}(u_0+d_1, v, w_0) \cdot \frac{\partial \mathbb{X}}{\partial v}(u_0+d_1, v, w_0)\, dv\right.\\
&\quad + \int_{w_0}^{w_0+d_3} \mathbb{f}(u_0+d_1, v_0+d_2, w) \cdot \frac{\partial \mathbb{X}}{\partial w}(u_0+d_1, v_0+d_2, w)\, dw\\
&\quad - \int_{v_0}^{v_0+d_2} \mathbb{f}(u_0+d_1, v, w_0+d_3) \cdot \frac{\partial \mathbb{X}}{\partial v}(u_0+d_1, v, w_0+d_3)\, dv\\
&\quad \left.- \int_{w_0}^{w_0+d_3} \mathbb{f}(u_0+d_1, v_0, w) \cdot \frac{\partial \mathbb{X}}{\partial w}(u_0+d_1, v_0, w)\, dw\right\}\\
&\quad + \left\{\int_{w_0}^{w_0+d_3} \mathbb{f}(u_0, v_0+d_2, w) \cdot \frac{\partial \mathbb{X}}{\partial w}(u_0, v_0+d_2, w)\, dw\right.\\
&\quad + \int_{u_0}^{u_0+d_1} \mathbb{f}(u, v_0+d_2, w_0+d_3) \cdot \frac{\partial \mathbb{X}}{\partial u}(u_0, v_0+d_2, w_0+d_3)\, dw\\
&\quad - \int_{w_0}^{w_0+d_3} \mathbb{f}(u_0+d_1, v_0+d_2, w) \cdot \frac{\partial \mathbb{X}}{\partial w}(u_0+d_1, v_0+d_2, w)\, dw\\
&\quad \left.- \int_{u_0}^{u_0+d_1} \mathbb{f}(u, v_0+d_2, w_0) \cdot \frac{\partial \mathbb{X}}{\partial u}(u_0, v_0+d_2, w_0)\, dw\right\}\\
&\quad - \left\{\int_{u_0}^{u_0+d_1} \mathbb{f}(u, v_0, w_0) \cdot \frac{\partial \mathbb{X}}{\partial u}(u, v_0, w_0)\, du\right.\\
&\quad + \int_{v_0}^{v_0+d_2} \mathbb{f}(u_0+d_1, v, w_0) \cdot \frac{\partial \mathbb{X}}{\partial v}(u_0+d_1, v, w_0)\, dv\\
&\quad - \int_{u_0}^{u_0+d_1} \mathbb{f}(u, v_0+d_2, w_0) \cdot \frac{\partial \mathbb{X}}{\partial u}(u, v_0+d_2, w_0)\, du
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{v_0}^{v_0+d_2} \mathbb{f}(u_0, v, w_0) \cdot \frac{\partial \mathbb{X}}{\partial v}(u_0, v, w_0)\, dv \Bigg\} \\
&- \Bigg\{ \int_{v_0}^{v_0+d_2} \mathbb{f}(u_0, v, w_0) \cdot \frac{\partial \mathbb{X}}{\partial v}(u_0, v, w_0)\, dv \\
&+ \int_{w_0}^{w_0+d_3} \mathbb{f}(u_0, v_0 + d_2, w) \cdot \frac{\partial \mathbb{X}}{\partial w}(u_0, v_0 + d_2, w)\, dw \\
&- \int_{v_0}^{v_0+d_2} \mathbb{f}(u_0, v, w_0 + d_3) \cdot \frac{\partial \mathbb{X}}{\partial v}(u_0, v, w_0 + d_3)\, dv \\
&- \int_{w_0}^{w_0+d_3} \mathbb{f}(u_0, v_0, w) \cdot \frac{\partial \mathbb{X}}{\partial w}(u_0, v_0, w)\, dw \Bigg\} \\
&- \Bigg\{ \int_{w_0}^{w_0+d_3} \mathbb{f}(u_0, v_0, w) \cdot \frac{\partial \mathbb{X}}{\partial w}(u_0, v_0, w)\, dw \\
&+ \int_{u_0}^{u_0+d_1} \mathbb{f}(u, v_0, w_0 + d_3) \cdot \frac{\partial \mathbb{X}}{\partial u}(u_0, v_0, w_0 + d_3)\, dw \\
&- \int_{w_0}^{w_0+d_3} \mathbb{f}(u_0 + d_1, v_0, w) \cdot \frac{\partial \mathbb{X}}{\partial w}(u_0 + d_1, v_0, w)\, dw \\
&- \int_{u_0}^{u_0+d_1} \mathbb{f}(u, v_0, w_0) \cdot \frac{\partial \mathbb{X}}{\partial u}(u_0, v_0, w_0)\, dw \Bigg\} \quad \text{[Theorem 4.1]} \\
&= 0 \qquad (5.2)
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
&\int_{u_0}^{u_0+d_1} \int_{v_0}^{v_0+d_2} \int_{w_0}^{w_0+d_3} \operatorname{div}(\operatorname{rot} \mathbb{f}) \left[\frac{\partial \mathbb{X}}{\partial u}, \frac{\partial \mathbb{X}}{\partial v}, \frac{\partial \mathbb{X}}{\partial w} \right] du\, dv\, dw \\
&\quad = d_1 d_2 d_3 (\operatorname{div}(\operatorname{rot} \mathbb{f}))(\mathbb{x}_0) \left[\frac{\partial \mathbb{X}}{\partial u}(\mathbb{u}_0), \frac{\partial \mathbb{X}}{\partial v}(\mathbb{u}_0), \frac{\partial \mathbb{X}}{\partial w}(\mathbb{u}_0) \right] \quad \text{[Proposition 2.1]} \qquad (5.3)
\end{aligned}
$$

Therefore, if we take $u = x$, $v = y$, and $w = z$, then it follows from (5.2) and (5.3) that (5.1) obtains. □

REFERENCES

Fukaya, K. (1995). *Elecromagnetic Fields and Vector Analysis*, Iwanami, Tokyo (in Japanese).
Hirota, R. (2000). *Lectures on Difference Equations*, Science, Tokyo (in Japanese).
Hsu, H. P. (1969). *Vector Analysis*, Simon and Schuster, New York.
Ihara, S. (1982). *Linear Spaces: I and II*, 2nd edn., Iwanami, Tokyo (in Japanese).

Kock, A. (1981). *Synthetic Differential Geometry*, (London Mathematical Society Lecture Note Series, vol. 51), Cambridge University Press, Cambridge.

Lavendhomme, R. (1996). *Basic Concepts of Synthetic Differential Geometry*, Kluwer, Dordrecht.

Moerdijk, I. and Reyes, G. E. (1991). *Models for Smooth Infinitesimal Analysis*, Springer, New York.

Nishimura, H. (1999). Infinitesimal calculus of variations. *International Journal of Theoretical Physics* **38**, 1771–1782.